Parabolic dynamics in the disk and in the ball

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Outline of my Talk

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Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$

n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}$

By **Schwarz's lemma**, f is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \overline{w} z} \right|$$

Theorem (Denjoy-Wolff)

If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.

if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$

if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

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The point p is called the **Denjoy-Wolff point** of f.

Cases:

1.$p \in \mathbb{D}$ f is called elliptic

2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic

3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

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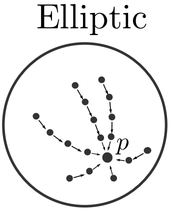

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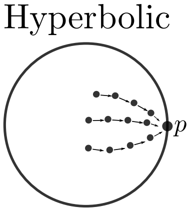

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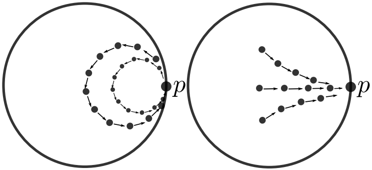

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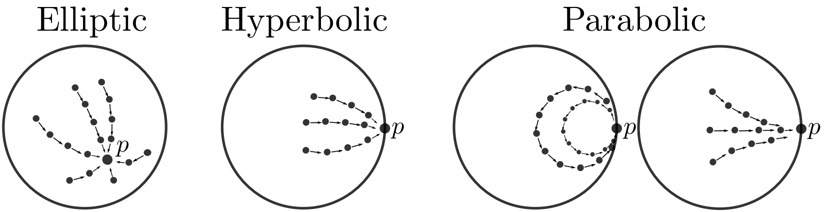

Parabolic case in the disk (or half-plane)

Consider a forward orbit

$$z_n = f_n(z_0) := \underbrace{f \circ \ldots \circ f}_{n \ times}(z_0)$$

By Schwarz's lemma $d(z_n, z_{n+1}) \leq d(z_{n-1}, z_n)$, and the **pseudo-hyperbolic step** $d_n := d(z_n, z_{n+1})$ must have a limit: $d_n \xrightarrow[n\to\infty]{} b$

Definition

We will call a sequence $\{z_n\}$ a **zero step** *(resp.* **non-zero step***) sequence if $b = 0$ (resp. $b > 0$).*

Another model: right-half plane $\mathbb{H} := \{z \mid \operatorname{Re} z > 0\}$, biholomorphically equivalent to the unit disk $\mathbb{D}$.

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Theorem (Pommerenke)

Consider f a parabolic self-map of $\mathbb{H}$ *with Denjoy-Wolff point* ∞*, and define* $z_n = x_n + iy_n := f_n(1)$*,*

$$g_n(z) := \frac{f_n(z) - iy_n}{x_n}.$$

Then the limit $g(z) = \lim_{n\to\infty} g_n(z)$ *exists locally uniformly and*

$$g(f(z)) = \phi(g(z)) \quad \forall z \in \mathbb{H},$$

and

- $\phi(z) = z + ib$ *(vertical translation) if* $\{z_n\}$ *has non-zero step;*
- $\phi(z) = z$ *and* $g(z) \equiv 1$ *(trivial conjugation) if* $\{z_n\}$ *has zero step.*

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Corollary 1.

The step does not depend on the choice of the sequence and depends on map only; i.e. for a given parabolic map either all orbits have zero step or all orbits have non-zero step.

Thus we can classify parabolic maps of the disk (or a half-plane) as **parabolic zero-step** and **parabolic non-zero step** maps.

Corollary 2.

In parabolic non-zero-step case in $\mathbb{H}$,

$$\arg z_n \xrightarrow[n\to\infty]{} \pm\frac{\pi}{2}$$

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Theorem (Baker, Pommerenke)

Let f be parabolic zero-step map of $\mathbb{H}$ with Denjoy-Wolff point infinity, then there exists $h : \mathbb{H} \to \mathbb{C}$ such that

$$h(f(z)) = h(z) + 1 \quad \forall z \in \mathbb{H},$$

i.e. f is conjugated to a horizontal shift in the plane.

Orbits in parabolic zero-step case may converge tangentially as well as non-tangentially.

Conjecture 1.

Let f be a parabolic zero-step map of $\mathbb{H}$ with Denjoy-Wolff point infinity, then there exists direction $\theta \in [-\pi/2, \pi/2]$ such that for any orbit $\{z_n\}$

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Multi-dimensional case

f is self-map of N-dimensional unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$.

Schwarz's lemma still holds in $\mathbb{B}^N$, with pseudo-hyperbolic distance defined as

$$d_{\mathbb{B}^N}(Z, W) := \left(\frac{|1 - \langle Z, W\rangle|^2}{\left(1 - \|Z\|^2\right)\left(1 - \|W\|^2\right)} \right)^{1/2}.$$

Multi-dimensional version of Denjoy-Wolff theorem holds:

Theorem (Hervé, MacCluer, 1983)

If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1]$ is a multiplier of f at p.

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An analog of the half-plane $\mathbb{H}$ in several dimensions is

Siegel domain (or Siegel half-space)

$$\mathbb{H}^N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \mathrm{Re}\, z > \|w\|^2\},$$

which is biholomorfically equivalent to $\mathbb{B}^N$ via

Cayley transform:

$\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$$\mathcal{C}((z, w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right) \quad \mathcal{C}^{-1}((z, w)) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right).$$

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For parabolic maps of the ball, zero and non-zero step cases are well-defined only **for sequences**.

The question whether the same map can have sequences of both types is still open.

Conjecture 2.

Let f a self map of $\mathbb{B}^N$ *of parabolic type. If the step* $d_{\mathbb{B}^N}(f_n(Z_0), f_{n+1}(Z_0)) \to 0$ *for some* $Z_0 \in \mathbb{B}^N$*, then* $d_{\mathbb{B}^N}(f_n(Z), f_{n+1}(Z)) \to 0$ *for all* $Z \in \mathbb{B}^N$*.*

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Definition

The **Koranyi region** $K(X, M)$ *of vertex* $X \in \partial\mathbb{B}^N$ *and amplitude* $M > 1$ *is the set*

$$K(X, M) = \left\{ Z \in \mathbb{B}^N \,\middle|\, \frac{|1 - \langle Z, X \rangle|}{1 - \|Z\|} < M \right\}.$$

When $N = 1$, it is the usual Stolz angle in the disk; but for $N > 1$ the region is tangent to the boundary of the ball along some directions.

Definition

For $X \in \partial\mathbb{B}^N$*, a sequence* $Z_n \to X$ *is called* **special** *if*

$$\lim_{n\to\infty} \frac{\|Z_n - \langle Z_n, X \rangle X\|^2}{1 - \|\langle Z_n, X \rangle X\|^2} = 0,$$

and **restricted** *if it is special and its orthogonal projection* $\langle Z_n, X \rangle X$ *is non-tangential.*

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non-tangential ⇒ restricted ⇒ lies in a Koranyi region

Theorem (O.O.)

If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ for parabolic self-map of the unit ball is restricted, then it must have zero step, i.e. $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$.

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Known examples

of parabolic maps in $\mathbb{H}^N$ are:

Example 1: Heisenberg translations

$(z, w) \longmapsto (z + z_0 + 2\langle w, w_0\rangle, w + w_0)$ *for some* $(z_0, w_0) \in \partial\mathbb{H}^N$, *i.e.* $\operatorname{Re} z_0 = \|w_0\|^2$.

They are parabolic automorphisms of $\mathbb{H}^N$ and thus have non-zero step.

Example 2: Generalized Heisenberg translations

$(z, w) \longmapsto (z + z_0 + 2\langle w, w_0\rangle, w + w_0)$ *with* $\operatorname{Re} z_0 \geq \|w_0\|^2$.

They have zero step unless $\operatorname{Re} z_0 = \|w_0\|^2$.

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Example 3: **Parabolic linear-fractional maps of $\mathbb{H}^N$**

(linear-fractional self-maps of the ball, transferred to $\mathbb{H}^N$).

$$f(Z) := \frac{AZ + B}{\left\langle Z, \overline{C} \right\rangle + d}$$

with $f(\mathbb{B}^N) \subseteq \mathbb{B}^N$, where A is $N \times N$-matrix, $B, C \in \mathbb{C}^N$ and $d \in \mathbb{C}$.

Theorem (Bayart)

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Example 4. (O.O.):

Given one-dimensional $\phi : \mathbb{H} \to \mathbb{H}$ of hyperbolic or parabolic type, with the Denjoy-Wolff point ∞,

construct $f(z,w) := (\phi(z - w^2) + w^2, w)$. Then:

f is the self-map of $\mathbb{H}^2$ with the Denjoy-Wolff point ∞ and has the same type and same multiplier at ∞ as ϕ.

Moreover, all forward orbits have zero (resp. non-zero) step, if ϕ is parabolic zero (resp. non-zero) step map.

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f *is the self-map of* $\mathbb{H}^2$ *with the Denjoy-Wolff point* ∞ *and has the same type and same multiplier at* ∞ *as* ϕ*.*
Moreover, all forward orbits have zero (resp. non-zero) step, if ϕ *is parabolic zero (resp. non-zero) step map.*

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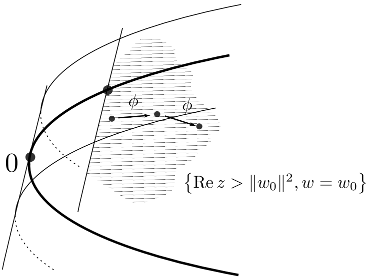

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Given one-dimensional $\phi : \mathbb{H} \to \mathbb{H}$ *of parabolic type, with the Denjoy-Wolff point* ∞*, construct*

$$f(z, w) := (\phi(z) + z_0 + 2\langle w, w_0 \rangle, w + w_0)$$

for some $(z_0, w_0) \in \partial \mathbb{H}^N$*.*

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Example 6. (Bayart) **Maps with some regularity at the Denjoy-Wolff point**

$D^{n+\varepsilon}$: parabolic maps of $\mathbb{B}^2$ that can be expanded near the Denjoy-Wolff point up to a certain order.

Depending on n and the first derivative matrix, they can be conjugated to various generalized Heisenberg translations, in particular:

- If $n = 5$ and the matrix is non-diagonalizable, model map is $(z, w) \longmapsto (z + z_0 + 2\langle w, w_0 \rangle, w + w_0)$
- If $n = 6$ and the matrix is diagonalizable, model map is $z \longmapsto z + b$

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Thank you!