Convergence of backward iteration sequences with bounded hyperbolic step in higher dimension

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One-dimensional case

Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$

n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n\ times}$

Theorem (Denjoy-Wolff)
If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \bar{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.

if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$

if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

The point p is called the **Denjoy-Wolff point** of f.

Cases:

1.$p \in \mathbb{D}$ f is called elliptic

2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic

3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

Corollary: every other fixed point q of f must lie on $\partial\mathbb{D}$, and its multiplier $f'(q) > 1$ (if finite).

If $q \in \partial\mathbb{D}$ and $f(q) = q$ and $1 < f'(q) < \infty$, then q is called **boundary repelling fixed point** (BRFP).

Backward-iteration sequence:
$\{w_n\}_{n=0}^{\infty}$, $f(w_{n+1}) = w_n$ for $n = 0, 1, 2 \ldots$

The pseudo-hyperbolic distance $\forall z, w \in \mathbb{D}$:

$$0 \le d_{\mathbb{D}}(z, w) = \frac{|z - w|}{|1 - \bar{z}w|} < 1$$

Theorem (Poggi-Corradini, 2003)
Let $\{w_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map of the disk f with bounded pseudo-hyperbolic step $d(w_n, w_{n+1}) \le a < 1$. Then:

1. $w_n \to q \in \partial\mathbb{D}$, and q is a fixed point with a well-defined multiplier $f'(q) < \infty$

2. If $q \ne p$, then q is a BRFP (i.e. $f'(q) > 1$). If $q = p$, f is of parabolic type.

3. When q is BRFP, the convergence $w_n \to q$ is non-tangential.

4. If $q = p$, then $w_n \to q$ tangentially.

Multi-dimensional case

$\mathbb{C}^N$, inner product $(z,w) = \sum_{j=1}^{N} z_j \bar{w}_j$

$\|z\|^2 = (z,z)$

Unit ball $\mathbb{B}^N = \{z \in \mathbb{C}^N : \|z\| < 1\}$

Multi-dimensional version of Denjoy-Wolff theorem holds:

Theorem (MacCluer, 1983)
If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{z \to p} \frac{1-\|f(z)\|}{1-\|z\|} \in (0,1]$ is a multiplier of f at p.

f is called hyperbolic if $c < 1$ and parabolic if $c = 1$.

Main Result

Theorem Let f be a holomorphic self-map of $\mathbb{B}^N$ of hyperbolic type (with Denjoy-Wolff point $p \in \partial\mathbb{B}^N$), $\{Z_n\}$ be a backward-itaration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:

1. There exists a point $\partial\mathbb{B}^N \ni \tau \neq p$ such that $Z_n \xrightarrow[n\to\infty]{} \tau$

2. $\{Z_n\}$ stays in a Koranyi region

3. Julia's lemma holds for τ with multiplier $A \geq \frac{1}{c}$, where c is the multiplier at p.

Horosphere of center $x \in \partial\mathbb{B}^N$ and radius $R > 0$:

$$E(x,R) = \left\{ z \in \mathbb{B}^N : \frac{|1-(z,x)|^2}{1-\|z\|^2} < R \right\}$$

Julia's lemma in $\mathbb{B}^N$:

Let f be a holomorphic self-map of $\mathbb{B}^N$ and $x \in \partial\mathbb{B}^N$ such that

$$\liminf_{z \to x} \frac{1-\|f(z)\|}{1-\|z\|} = \alpha < \infty$$

Then there exists a unique $y \in \partial\mathbb{B}^N$ such that $\forall R > 0 \; f(E(x,R)) \subset E(y, \alpha R)$.

Siegel domain:

$\mathbb{H}^N = \{(z,w) \in \mathbb{C} \times \mathbb{C}^{N-1} : Rez > \|w\|^2\}$

Cayley transform: $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$$\mathcal{C}((z,w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z} \right)$$

$$\mathcal{C}^{-1}((z,w)) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1} \right)$$

WLOG let Denjoy-Wolff point be $\infty \in \mathbb{H}^N$, $Z_n = (t_n, 0)$, $Z_{n+1} = (z, w)$

Then $E(\infty, t) = \{Rez - \|w\|^2 > t\}$

$t_n := Rez_n - \|w_n\|^2$

$$\begin{cases} Rez - \|w\|^2 \leq ct_n \\ \\ \left|\dfrac{z - t_n}{z + t_n}\right|^2 + \dfrac{4t_n\|w\|^2}{|z + t_n|^2} \leq a^2 \end{cases}$$

Claim $\|pr(Z_n) - pr(Z_{n+1})\|^2 \leq Ct_n$

Using $t_{n+k} \leq c^k t_n$, $k = 1, 2, \ldots$ (Julia's lemma)
$\Longrightarrow \quad pr(Z_n) \xrightarrow[n \to \infty]{} \tau \in \partial\mathbb{H}^N \quad \Longrightarrow \quad Z_n \xrightarrow[n \to \infty]{} \tau$

Koranyi region (generalization of Stolz angle) of vertex $x \in \partial\mathbb{B}^n$ and amplitude $M > 1$:

$$K(x, M) = \left\{ z \in \mathbb{B}^N : \frac{|1 - (z, x)|}{1 - \|z\|} \leq M \right\}$$

Intersection of $K((1, 0), M)$ with 1-dimensional complex subspace generated by $(1, 0)$ is

$$\left\{ z_1 \in \mathbb{D} : \frac{|1 - z_1|}{1 - |z_1|} \leq M \right\}$$ (usual Stolz region)

Intersection with (2n-1)-dimensional real space $\{z : Imz_1 = 0\}$ contains

$$(Rez_1 - 1/M)^2 + \|z'\|^2 < (1 - 1/M)^2$$

Open questions

1. Number of BRFP

2. Conjugation at BRFP

3. Parabolic and "elliptic" cases