Backward-iteration sequences and boundary repelling fixed points in higher dimension

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One-dimensional case, forward iteration

Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$
n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n\ times}$
By **Schwarz's lemma**, f is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left|\frac{z - w}{1 - \overline{w}z}\right|$$

Theorem (Denjoy-Wolff): If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.
if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$
if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits
The point p is called the **Denjoy-Wolff point** of f.
Cases:
1.$p \in \mathbb{D}$ f is called elliptic
2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic
3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

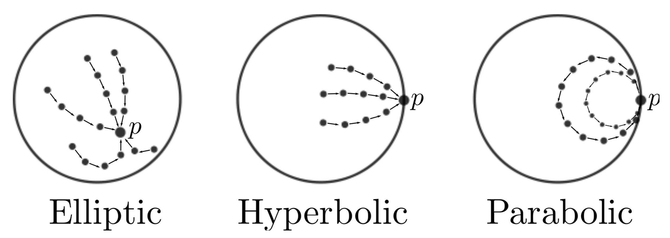

Figure 1: **Orbits near the Denjoy-Wolff point p.**

If $p \in \partial\mathbb{D}$, **Julia's lemma** holds for the point p, and multiplier $c = f'(p) \leq 1$:

$$\forall R > 0 \ \ f(H(p, R)) \subseteq H(p, cR),$$

where $H(p, R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R :

$$H(p, R) := \left\{z \in \mathbb{D} : \frac{|p - z|^2}{1 - |z|^2} < R\right\}$$

Conjugation in hyperbolic case:
Theorem (Valiron): There is an analytic map $\psi : \mathbb{D} \to \mathbb{H}$ (where $\mathbb{H}$ is the right half-plane), which solves the Schröder equation:

$$\psi \circ f = \frac{1}{c}\psi,$$

and so ψ conjugates f to multiplication in $\mathbb{H}$

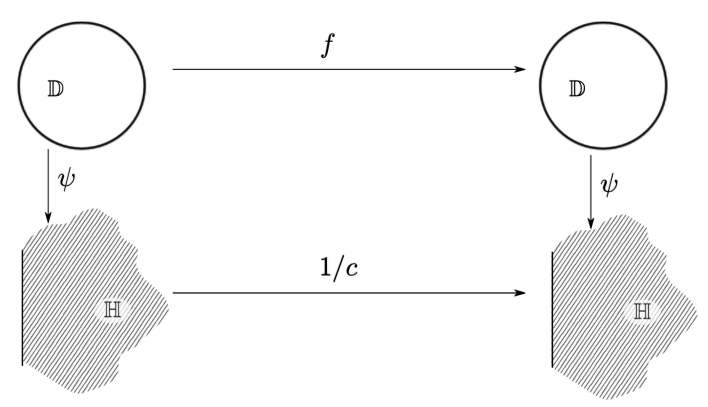

Figure 2: **Conjugation for forward iteration.**

One-dimensional case, backward iteration

Backward-iteration sequence:
$\{z_n\}_{n=0}^{\infty}$, $f(z_{n+1}) = z_n$ for $n = 0, 1, 2 \ldots$
The sequence $d(z_n, z_{n+1})$ is increasing, so we need a bound on the pseudo-hyperbolic step:

$$d(z_n, z_{n+1}) \leq a < 1$$

Theorem (Poggi-Corradini): Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map of the disk f with bounded pseudo-hyperbolic step $d(z_n, z_{n+1}) \leq a < 1$. Then:
1. $z_n \to q \in \partial\mathbb{D}$, and q is a fixed point with a well-defined multiplier $f'(q) < \infty$
2. If $q \neq p$, then q is a **boundary repelling fixed point** (BRFP) (i.e. $f'(q) > 1$). If $q = p$, f is of parabolic type.
3. When q is BRFP, the convergence $z_n \to q$ is non-tangential.
4. If $q = p$, then $z_n \to q$ tangentially.

Conjugation:
Theorem (Poggi-Corradini): an analytic self-map of the unit disc $\mathbb{D}$ f with BRFP $1 \in \partial\mathbb{D}$ and multiplier α at 1 can be conjugated to the automorphism $\eta(z) = (z - a)/(1 - az)$, where $a = (\alpha - 1)/(\alpha + 1)$:

$$\psi \circ \eta(z) = f \circ \psi(z),$$

via an analytic map ψ of $\mathbb{D}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$, which has non-tangential limit 1 at 1.

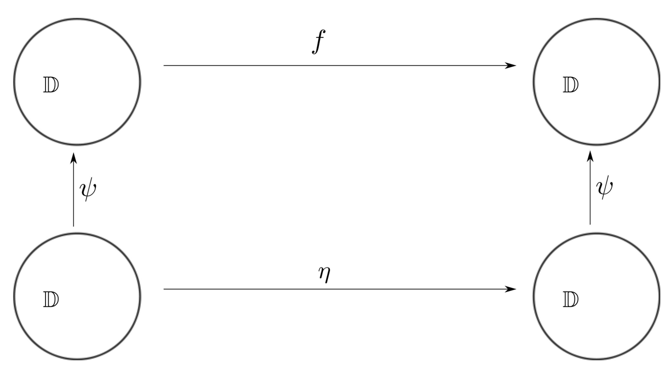

Figure 3: **Conjugation for backward iteration.**

N-dimensional case, forward iteration

Consider $\mathbb{C}^N$, inner product $(Z, W) = \sum_{j=1}^{N} Z_j\overline{W_j}$
$\|Z\|^2 = (Z, Z)$
Unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$
Julia's lemma in $\mathbb{B}^N$: Let f be a holomorphic self-map of $\mathbb{B}^N$ and $X \in \partial\mathbb{B}^N$ such that

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that $\forall R > 0$ $f(H(X, R)) \subset H(Y, \alpha R)$.
Horosphere of center $X \in \partial\mathbb{B}^N$ and radius $R > 0$:

$$H(X, R) = \left\{Z \in \mathbb{B}^N : \frac{|1 - (Z, X)|^2}{1 - \|Z\|^2} < R\right\}$$

Theorem (MacCluer): If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1]$ is a multiplier of f at p.

f is called hyperbolic if $c < 1$ and parabolic if $c = 1$.
Siegel domain:

$$\mathbb{H}^N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : Re z > \|w\|^2\}$$

is biholomorphic to $\mathbb{B}^N$ via Cayley transform: $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$$\mathcal{C}((z, w)) = \left(\frac{1 + z}{1 - z}, \frac{w}{1 - z}\right)$$

$$\mathcal{C}^{-1}((z, w)) = \left(\frac{z - 1}{z + 1}, \frac{2w}{z + 1}\right)$$

Conjugation results:
- (Bracci, Gentili, Poggi-Corradini): conjugation to a multiplication via $\psi : \mathbb{B}^N \to \mathbb{H}$.
- (Bracci, Gentili): f is conjugated to its linear part, assuming some regularity at the Denjoy-Wolff point.

N-dimensional case, backward iteration

Theorem 1. Let f be a analytic self-map of $\mathbb{B}^N$ of hyperbolic type (with Denjoy-Wolff point $p \in \partial\mathbb{B}^N$), $\{Z_n\}$ be a backward-iteration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:
1. There exists a point $\partial\mathbb{B}^N \ni \tau \neq p$ such that $Z_n \xrightarrow[n \to \infty]{} \tau$
2. $\{Z_n\}$ stays in a Koranyi region
3. Julia's lemma holds for τ with multiplier $\alpha \geq \frac{1}{c}$, where c is the multiplier at p.
Since $\alpha \geq \frac{1}{c} > 1$, the point $q \in \partial\mathbb{B}^N$ is called the **boundary repelling fixed point** for f.
Characterization of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.
Theorem 2. Suppose $f : \mathbb{H}^N \to \mathbb{H}^N$ is an analytic function of hyperbolic type and 0 is an isolated boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$. Then f is conjugated to the automorphism $\eta(z, w) = (\alpha z, \sqrt{\alpha}w)$

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map ψ.

Construction of ψ: $\psi = \lim_{n \to \infty} \{f_n \circ \tau_n \circ p_1\}$,
where $p_1(z, w) := (z, 0)$ is the projection on the first (radial) dimension, so

$$\psi(z, w) = \psi(z, 0)$$

and is essentially one-dimensional map.

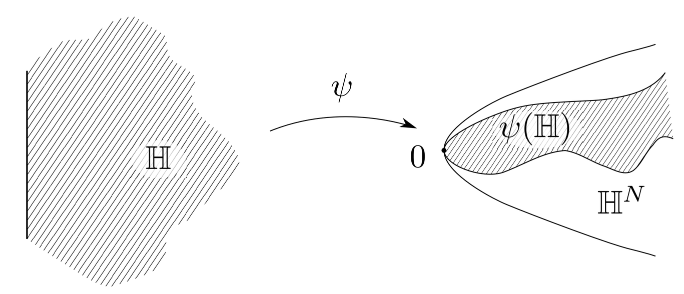

Figure 4: **The image of ψ in Siegel domain.**

An analytic map $f : \mathbb{H}^N \to \mathbb{H}^N$ is called **expandable** at 0 if

$$f(z, w) = (\alpha z + o(|z|), Aw + o(|z|^{1/2})).$$

In particular, 0 is a fixed point of f and α is the multiplier of f at 0.
Theorem 3. Let f be expandable at 0, of hyperbolic type, and let the matrix A be diagonal, and WLOG
$|a_{j,j}| = \sqrt{\alpha}$ for $j = 1 \ldots L$
$|a_{j,j}| < \sqrt{\alpha}$ for $j = L + 1 \ldots N - 1$.
Then f is conjugated to the automorphism $\eta(z, w) = (\alpha z, \Omega\sqrt{\alpha}w)$ (Ω is a rotation):

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map $\psi(z, w) = \psi(p_L(z, w))$, where p_L is a projection on the first $L + 1$ dimensions.