DISCRETE DYNAMICS OF CONTRACTIONS ON GRAPHS

OLENA OSTAPYUK AND MARK RONNENBERG

ABSTRACT. We study the dynamical behavior of functions on vertices of a graph that are contractions in the graph metric. We show that the fixed point set of such functions must be convex. If a function has no fixed points and the graph is a tree, we prove that every dynamical cycle must have an even period and the function behaves eventually like a symmetry.

1. INTRODUCTION

This work was inspired by dynamics of analytic functions on the unit disk. The key property of such functions is the point-invariant Schwarz Lemma, i.e. that analytic functions are contractions in the hyperbolic metric of the disk. This property allows to prove various results about iteration of analytic functions, see, for example, survey paper [4].

Our purpose is to study dynamics of contractions in a discrete setting. In particular we study dynamics on finite graphs (in most cases, trees). A connected graph can be considered as a discrete metric space of vertices with the *graph metric*. Let $G = (V, E)$ be a finite, connected, simple graph with the set of vertices V and the set of edges E. Then for all vertices $x, y \in V$ we say the distance between x and y, denoted $d(x, y)$, is the number of edges in the shortest path connecting x to y. Such path is called a *geodesic*. Note that trees as metric spaces are 0-hyperbolic ([2]), so we expect them to have some similar properties to the unit disk with hyperbolic metric.

We wish to study contractions (in the graph metric) on the vertices of a graph. Let f be a function on the vertices of G to the vertices of G. We say f is a *contraction* if for all vertices $x, y \in V$ $d(f(x), f(y)) \leq d(x, y)$. We will need some terminology from dynamics. Let f be a function. We denote by $f^{\circ n}(x) = f \circ f \circ f \circ \ldots \circ f(x)$ (n terms) the n^{th} *iterate* of f. If for some point x and some positive integer n $f^{\circ n}(x) = x$, then we say x is periodic point, x lies on a *dynamical cycle* of f of period n, or that x lies on a *dynamical n-cycle* of f. If $f(x) = x$, we say x is a *fixed point* of f. We use the term *dynamical cycle* to distinguish these cycles from the graph cycles.

It is easy to show by induction that, given a contraction f, $f^{\circ n}$ is also a contraction for any positive integer n. Dynamical cycles and fixed points will be the main focus of our study.

2. FIXED POINT SETS

Our goal is to characterize the set of fixed points of a contraction on graph vertices. Note that in the general case, the fixed point set can be empty:

Example 2.1. Let G_1 be a graph with four vertices x, y, z, w. Let f be a function on the vertices of G_1 defined by $f(x) = y$, $f(y) = z$, $f(z) = w$, and $f(w) = x$. Then

$\{x, y, z, w\}$ forms a dynamical 4-cycle of f (see Figure 1). f is clearly a contraction since for all $a, b \in \{x, y, w, z\}$, we have $d(f(a), f(b)) = d(a, b)$. In this case, the set of fixed points of f is empty.

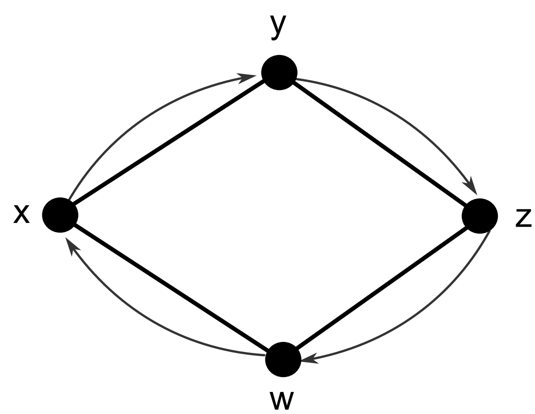

FIGURE 1. Dynamical 4-cycle.

Example 2.2. Let G_2 be a graph with vertices x_0, x_1, x_2, y_0, y_1, z_0 and z_1 (see Figure 2). Let f be a contraction on the vertices of G_2 such that x_0, x_1, x_2 are fixed by f, and $\{y_0, y_1\}$ and $\{z_0, z_1\}$ are dynamical 2-cycles of f.

Note that one main difference between the two examples is that for any two vertices in G_2, the geodesic connecting them is unique, whereas this is not the case with G_1. Notice also that for any two fixed points in G_2, the geodesic connecting them contains only fixed points.

Definition 2.3. Let $G = (V, E)$ be a graph and let $H \subset V$. We say H is *convex* if for any two vertices in H the geodesic connecting them contains only vertices in H. (See, for example, [3].)

Thus in Example 2.2, the set of fixed points of f is convex. In fact, this is true in general:

Theorem 2.4. *Let $G = (V, E)$ be a graph such that the geodesic between any two vertices is unique. Let f be a contraction on the vertices of G. Then the set of fixed points of f is convex.*

Proof. Let $x, y \in V$ be fixed by f. Let L be the unique geodesic connecting them. Let $z \in L$. We need to show that $f(z) = z$. We will first show that $f(z) \in L$ and it will follow that $f(z) = z$.

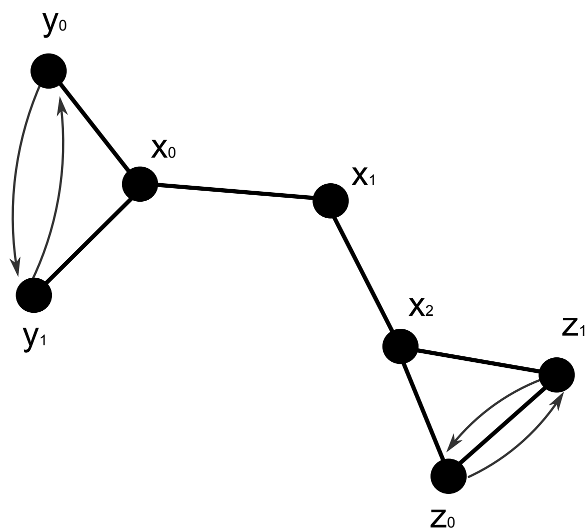

FIGURE 2. Cycles and fixed points.

By way of contradiction, suppose that $f(z) \notin L$. Then there exist unique geodesics connecting x to $f(z)$ and y to $f(z)$, respectively. We can concatenate these geodesics to construct a walk K connecting x to y. Then the length of K is $d(x, f(z)) + d(f(z), y)$ and the length of L is $d(x, z) + d(z, y)$. Since f is a contraction and x and y are fixed points, we have $d(f(z), x) \leq d(z, x)$ and $d(f(z), y) \leq d(z, y)$. Then it follows that $d(x, f(z)) + d(f(z), y) \leq d(x, z) + d(z, y)$. If $d(x, f(z)) + d(f(z), y) = d(x, z) + d(z, y)$, then L is not a unique geodesic between x and y, a contradiction. If $d(x, f(z)) + d(f(z), y) < d(x, z) + d(z, y)$, then K is shorter than L, which is also a contradiction. Thus it must be that $z \in L$.

Now we will show that $f(z) = z$. Suppose $f(z) \neq z$. Since $f(z)$ lies on the geodesic L connecting x to y, we have $d(x, z) + d(z, y) = d(x, f(z)) + d(f(z), y) = d(x, y)$. We can assume without loss of generality that $d(x, f(z)) < d(x, z)$, in which case we obtain $d(y, f(z)) = d(x, y) - d(x, f(z)) > d(x, y) - d(x, z) = d(y, z)$, contradicting the fact that f is a contraction. Thus we conclude that $f(z) = z$.

□

Note that if for any two points in G the geodesic connecting them is not unique then the conclusion of Theorem 2.4 does not necessarily hold, as can be seen in the following counterexample.

Example 2.5. Let G_3 be a graph with vertices x_0, x_1, y and z as shown in Figure 3.

Let f be a contraction such that the vertices z and y are fixed and the points x_0 and x_1 form a dynamical 2-cycle. Note that the geodesic connecting z to y is not unique, since the path from z to y through x_0 is the same length as the path through x_1. Despite the fact that z and y are fixed and that x_0, x_1 lie on the

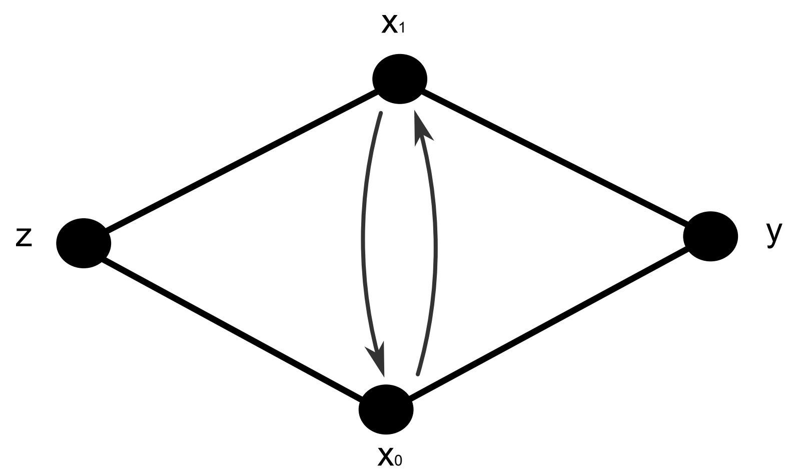

FIGURE 3. The graph G_3 with non-unique geodesics.

geodesics connecting them, x_0 and x_1 are clearly not fixed. Thus the conclusion of Theorem 2.4 does not hold in this case.

Corollary 2.6. *Let $G = (V, E)$ be a graph such that for any two vertices in G the geodesic connecting them is unique. Let f be a contraction on V. Suppose f has a dynamical cycle J of period k. Let z be a point which lies on the geodesic connecting two consecutive points in J. Then z lies on a dynamical cycle whose period divides k.*

Proof. Let $x, y \in J$. Let $z \in V$ such that z lies on the geodesic between x and y. Since J is a dynamical cycle of period k, $f^{\circ k}(x) = x$ and $f^{\circ k}(y) = y$. Thus x and y are fixed by the k^{th} iterate of f. Since f is a contraction, any iterate of f is also a contraction. Thus Theorem 2.4, applied to $f^{\circ k}$ implies that $f^{\circ k}(z) = z$. So z must lie on a dynamical cycle whose period divides k.

□

Now we will consider a particular case when the graph is a tree. For any tree, a path connecting any two points is unique, hence geodesics are unique, so Theorem 2.4 holds. But the converse is also true for trees: any convex set of vertices will be a fixed point set for some contraction.

We will need the following property of a tree structure: in a tree, a concatenation of two geodesics from x to y and from y to z is either a geodesic from x to z or a walk that follows the geodesic connecting x to y until the fist common point of two geodesics y', then follows the geodesic from y' to y, then goes back to y' along the same geodesic and finally follows the geodesic from y' to z. Note that concatenation of geodesics from x to y' and from y' to z will form a geodesic that connects x to z.

Proposition 2.7. *Let $T = (V, E)$ be a tree and $H \subset V$ be convex set. Then there exists a contraction f such that H is the fixed point set of f.*

Proof. Given H, we define the desired contraction f as follows: for all $x \in V$, $f(x) = y$, where $y \in H$ is the closest vertex to x in H. Note that such y is unique. If $x \notin H$ and there are two points y_1 and y_2 in H within the same (shortest) distance to x, we can consider the point x' on the intersection of the geodesics connecting x to y_1 and x to y_2 that the geodesics from y_1 to y_2 passes through it. Then this point is in H and also closer to x than either y_1 or y_2, which is contradiction. Thus the point y is unique and function f is well defined. Also, H is clearly fixed point set of f.

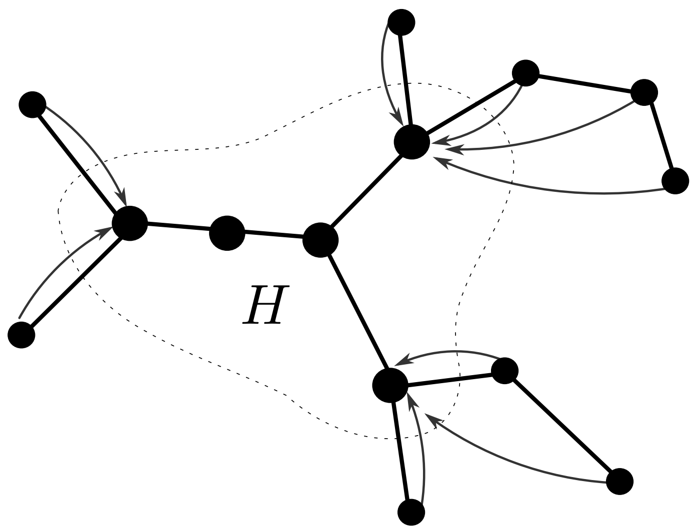

FIGURE 4. Constructing contraction f for a given convex subset of vertices H.

Now we need to show that f is a contraction. Let $f(x_1) = y_1$ and $f(x_2) = y_2$. Consider a walk following the geodesic from x_1 to y_1, then from y_1 to y_2. If there is common point of these geodesics other than y_1, then this point is in H and within shorter distance to x_1 than y_1, which contradicts the construction of y_1. So the concatenation of these two geodesics is the geodesic from x_1 to y_2. Similarly, the geodesic from y_1 to x_2 passes through y_2, and finally, the geodesic from x_1 to x_2 is just a concatenation of those from x_1 to y_1, y_1 to y_2 and y_2 to x_1. So we have

$$d(x_1, x_2) = d(x_1, y_1) + (dy_1, y_2) + d(y_2, x_2) \geq d(y_1, y_2),$$

and f is a contraction.

□

3. Contractions with No Fixed Points

In the previous section, we have characterized the set of fixed points of a contraction on the vertices of a graph with unique geodesics, in particular a tree. Next we want to consider the case when a contraction has no fixed points. Then there must exist a dynamical cycle. We will use the following property of periodic points:

Lemma 3.1. *Let G be a finite graph, and f be a contraction on vertices of G. If x and y are two periodic points of f (not necessarily from the same dynamical cycle), then $d(f(x), f(y)) = d(x, y)$.*

Proof. Assume x belongs to a dynamical m-cycle and y belongs to a dynamical n-cycle. Let K be a common multiple of m and n. Then we have:

$$d(x, y) \geq d(f(x), f(y)) \geq \ldots \geq d(f^{\circ K}(x), f^{\circ K}(y)) = d(x, y).$$

So all inequalities must be in fact equalities and in particular, $d(f(x), f(y)) = d(x, y)$. □

Now let us introduce some notations. Let $G = (V, E)$ be a graph and f a contraction on V. Let $J \subset V$ be a dynamical cycle of f. Then we denote by J' the set of all vertices which lie on geodesics connecting consecutive points in J, together with the vertices in J.

Theorem 3.2. *Let T be a finite tree. Let f be a contraction on the vertices of T. If f has no fixed points, then f has a dynamical 2-cycle such that the points in the cycle are connected by an edge. Moreover, such a cycle is unique.*

Proof. Suppose f has no fixed points. Since the number of vertices of T is finite, every vertex of T either lies on a dynamical cycle of period greater than 1 or is eventually mapped into one. Let k be the least period of all dynamical cycles of f. Let J be a dynamical cycle of period k such that the distance between consecutive points in J is least among all dynamical cycles of f of period k. We want to show that $k = 2$.

We claim that for $k > 2$ there must exist two geodesics connecting consecutive points in J that intersect at a point other than their end-points. If not the points in J' would form a graph cycle, which a contradiction since T is a tree. Thus there must exist two geodesics which intersect at a point which is not one of their endpoints.

Suppose two non-consecutive geodesics intersect at some point y. Then we claim that there must exist two consecutive geodesics which intersect at point z which is not one of their endpoints. Indeed, if we start from the point y of intersection of two non-consecutive geodesics and follow one of the geodesics to the point x_j on the cycle J, then follow the next geodesic to the point $x_{j+1} = f(x_j)$, and so on, we will eventually return to the point y. Since the graph is a tree, the walk constructed this way must go over each edge in this walk at least twice. In particular, there must exist a vertex w which is farthest away from y on this walk and an edge $\{w, z\}$ such that our walk will follow the edge from z to w and then immediately return to z through the same edge. Note that w must be an endpoint of two consecutive geodesics, because one geodesic cannot follow the same edge twice. Then z lies on the intersection of two consecutive geodesics.

Without loss of generality, let x_0, x_1, x_2 be the endpoints of the two consecutive geodesics constructed above. By Corollary 2.6, z must lie on a dynamical cycle whose period divides k, but since k is the least possible cycle length, z must lie on a dynamical k-cycle.

Since f is a contraction and x_0, x_1, x_2 are points on a dynamical cycle, f must map the geodesic from x_0 to x_1 one-to-one onto the geodesic from x_1 to x_2. Since z lies on the geodesic from x_0 to x_1, $f(z)$ must lie on the geodesic from x_1 to x_2. Thus both z and $f(z)$ lie on the geodesic from x_1 to x_2 and we have $d(z, f(z)) < d(x_1, x_2) = d(x_0, x_1)$. So we have found a dynamical k-cycle $\{z, f(z), \dots, f^{\circ(k-1)}(z)\}$ such that the distance between two consecutive point in this cycle is less that $d(x_0, x_1)$.

This contradicts the way we select J, so k must be equal to 2 and the geodesic from x_0 to x_1, which is the same as the geodesic from x_1 to x_0, must contain no other points. This means there is a dynamical 2-cycle $\{x_0, x_1\}$ and x_0 and x_1 are connected by an edge.

Now we need prove that such a dynamical 2-cycle is unique. Let $\{y_0, y_1\}$ be another such cycle. Without loss of generality assume that the distance a between x_0 and y_0 is the shortest among all distances from a point in $\{x_0, x_1\}$ to a point in $\{y_0, y_1\}$. Now consider x_1, it is connected to x_0 by an edge. If x_1 lies on the geodesic from x_0 to y_0, then $d(x_1, y_0) < d(x_0, y_0)$, which contradicts the choice of x_0, y_0. Otherwise, the geodesic from y_0 to x_1 follows the geodesic from y_0 to x_0 and then the edge connecting x_0 to x_1, so $d(y_0, x_1) = a+1$. Similarly, $d(x_0, y_1) = a+1$, and finally, $d(x_1, y_1) = a + 2$. But then $d(x_1, y_1) = d(f(x_0), f(y_0)) > d(x_0, y_0)$, which contradicts the assumption that f is a contraction.

□

It will in fact turn out that every dynamical cycle of a contraction with no fixed points has even period. To prove this we will need the following corollary to Theorem 3.2. Let us introduce the following notation. Let $\{x_0, x_1\}$ be the points in the 2-cycle, constructed in Theorem 3.2. We let X_0 denote the set of all points which are within shorter distance to x_0 than to x_1. Similarly we let X_1 denote the set of all points which are within shorter distance to x_1 than to x_0, see Figure 5.

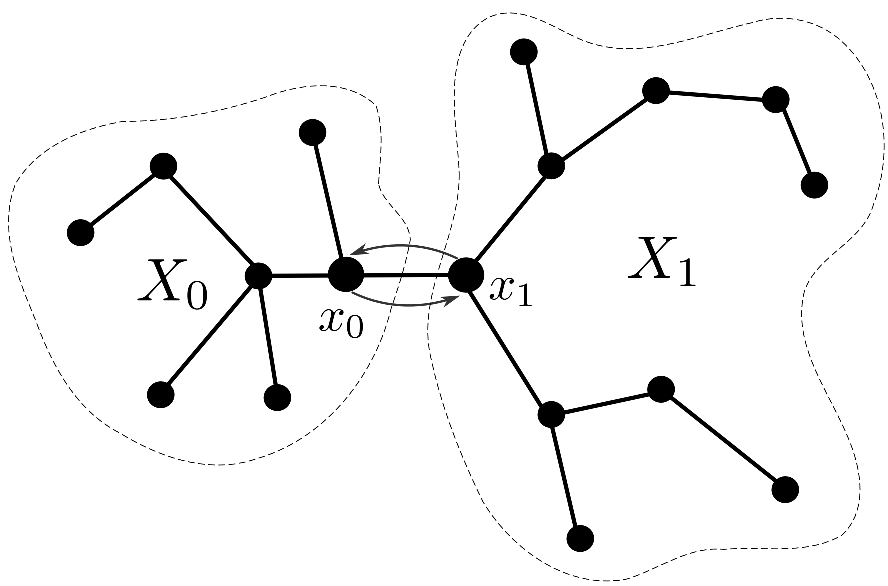

FIGURE 5. Unique 2-cycle $\{x_0, x_1\}$ and sets X_0 and X_1.

Corollary 3.3. *Let T be a finite tree and f a contraction on the vertices of T such that f has no fixed points. Let $\{x_0, x_1\}$ be the unique dynamical 2-cycle, where x_0*

and x_1 are connected by an edge. Then for all vertices z that lie on any dynamical cycle, if $z \in X_0$ (respectively X_1), then $f(z) \in X_1$ (respectively X_0).

Proof. Let z lie on a dynamical cycle and $z \in X_0$. By way of contradiction suppose that $f(z) \in X_0$. Let $a = d(z, x_0)$, then $d(z, x_1) = a + 1$. By Lemma 3.1, $d(f(z), x_1) = a$, and since $f(z) \in X_0$, we must have $d(f(z), x_0) < d(f(z), x_1) = a$. But by Lemma 3.1 again, $d(f(z), x_0) = d(f(z), f(x_1)) = d(z, x_1) = a + 1$, which is a contradiction. So $f(z) \in X_1$.

□

Note that if z is not a periodic point, then the above claim does not hold.

Example 3.4. Let T be a tree with vertices x_0, x_1 and z, such that there are edges between x_0 and x_1 and between x_0 and z, and f be a contraction such that $\{x_0, x_1\}$ form a dynamical 2-cycle and $f(z) = x_0$ (see Figure 6). Then f has no fixed points, and x_0 and x_1 form the unique 2-cycle connected by an edge. Since $f(z) = x_0$, we have $z \in X_0$ and also $f(z) \in X_0$. Thus we see that if a point z is in X_0 but does not lie on a dynamical cycle, it is not necessarily true that $f(z) \in X_1$.

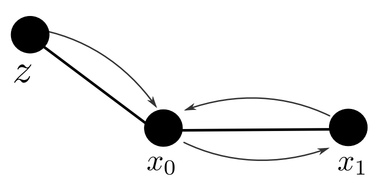

FIGURE 6

Now we are ready to prove the following:

Theorem 3.5. *Let T be a finite tree and f a contraction on the vertices of T such that f has no fixed points. Then every dynamical cycle of f has even period.*

Proof. Since f has no fixed points, f has a dynamical 2-cycle $\{x_0, x_1\}$ whose points are connected by an edge and sets of vertices X_0 and X_1 as defined above. Let $\{y_0, y_1, ..., y_{n-1}\}$ be a dynamical n-cycle of f. Without loss of generality, suppose $y_0 \in X_0$. Then by Corollary 3.3, we have $y_1 \in X_1$, and in general, $y_{2k} \in X_0$ and $y_{2k+1} \in X_1$. If n is odd, then $y_0 = f(y_{n-1}) \in X_1$, which is contradiction to $y_0 \in X_0$. Hence every dynamical cycle of f has even period.

□

If a contraction f on the vertices of a tree T has no fixed points, then f eventually behaves like a symmetry. More precisely:

Theorem 3.6. *Let $T = (V, E)$ be a finite tree and f a contraction on V without fixed points. Then there exists a subset H of V and a non-negative integer N such that $f^{\circ N}(V) = H$ and f is a symmetry on the connected subgraph induced by H. In particular, there is an edge in the subgraph, such that two connected components obtained by removing this edge are isomorphic graphs and f is an isomorphism.*

Proof. Since T is finite and has no fixed points, each vertex of T will be mapped eventually to a point on a dynamical cycle. Thus there exists N such that $f^{\circ N}(V) = H$ contains only periodic points of f. Note that by Corollary 2.6, the subgraph induced by H is connected. Let $\{x_0, x_1\}$ be the unique dynamical 2-cycle whose points are connected by an edge. Then by Corollary 3.3, for all $z \in H \cap X_0$, $f(z) \in H \cap X_1$ and for all $z \in H \cap X_1$, $f(z) \in H \cap X_0$. Moreover, since all points in H are periodic, f maps $H \cap X_0$ one-to-one and onto $H \cap X_1$. Now we need to show that any two vertices y, z in $H \cap X_0$ are connected by an edge if and only if $f(y)$ and $f(z)$ are connected by an edge. But being connected by an edge is equivalent to $d(y, z) = 1$, and since by Lemma 3.1, $d(y, z) = d(f(y), f(z))$, the required conclusion follows.

□

4. Conclusion

Note that in the classical case of the unit disk in the complex plane, any analytic self-map of the disk always has a fixed point in the closed disk. This is the consequence of the classical Denjoy-Wolff theorem (see, for example, [1] and references therein). In our study, a contraction without fixed points must behave like a symmetry. Symmetries are contractions in the unit disk, but they are not analytic (in fact, they are anticonformal, i.e. they preserve the value of angles, but change their orientation). So we can say that our result agrees with the classical case.

References

1. M. Abate, *Iteration theory of holomorphic maps on taut manifolds*, Mediterranean Press (1989).
2. J. Anderson, *Hyperbolic geometry*, Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London (1999).
3. Jonathan L. Gross and Jay Yellen, *Graph theory and applications*, Chapman and Hall/CRC, Boca Raton, FL (2006).
4. P. Poggi-Corradini, *Iteration in the disk and the ball: a survey of the role of hyperbolic geometry*, Anal. Math. Phys. (1 (2011), no. 4, 289-310).

Department of Mathematics, University of Northern Iowa, Cedar Falls, Iowa 50614-0506

E-mail address: ostapyuk@math.uni.edu

Department of Mathematics, University of Northern Iowa, Cedar Falls, Iowa 50614-0506

E-mail address: ronnemaa@uni.edu