THE LOEWNER PROPERTY

Abstract. Notes for the talk given at the Workshop on Carpets, CLE and Dessins, Seattle, August 6-11, 2012, on Mario Bonk's paper Uniformization of Sierpinski carpets in the plane, Invent. math. 186 (2011), 559-665.

Notations

- $\sigma(x, y)$ is the chordal distance on $\mathbb{C}_\infty$.
- Σ is the spherical measure on $\mathbb{C}_\infty$.
- $\Delta(E, F) := \dfrac{\mathrm{dist}(E, F)}{\min(\mathrm{diam}(E), \mathrm{diam}(F))}$ is the relative distance between sets E and F.
- $B(x, r) := \{y : \sigma(y, x) < r\}$ is the open ball of radius r centered at x.
- $N_\delta(E) := \{x : \mathrm{dist}(x, E) < \delta\}$ is the δ-neighborhood of E.
- $\Gamma(E, F; \Omega)$ is the family of all closed paths in Ω that connect E and F.
- $\mod(\Gamma)$ is the modulus of the path family Γ.

Definition. A region $\Omega \subseteq \mathbb{C}_\infty$ is called ϕ-Loewner if there exists a non-increasing $\phi : (0, \infty) \to (0, \infty)$ such that

$$\mathrm{Mod}(\Gamma(E, F; \Omega)) \geq \phi(\Delta(E, F)),$$

for any two disjoint continua E and F in $\overline{\Omega}$.

Equivalent to: Exists $m = m(t) > 0$ such that if $\Delta(E, F) \leq t$ and $\int \rho d\Sigma < m$ then there exists $\gamma \in \Gamma(E, F; \Omega)$ such that $\int_\gamma \rho ds < 1$. (Can take $m(t) := \phi(t)$ for ϕ-Loewner region and $\phi(s) := sup\{m(t) : t \geq s\}$.)

1. The unit disk $\mathbb{D}$ is Loewner. An annulus $N_\delta(\mathbb{D})\backslash\overline{\mathbb{D}}$ with $\delta \in (0, \sqrt{2})$ is ϕ-Loewner with $\phi = \phi(\delta)$.

2. The image of a Loewner region under quasi-Möbius map is Loewner.

3. A Jordan region bounded by a quasicircle is Loewner.

4. Collar lemma: For a region $\Omega = \mathbb{C}_\infty \backslash \bigcup_{i=1}^n D_i$ with complementary components D_i being s-relatively separated closed Jordan regions with boundaries ∂D_i being k-quasicircles, it is possible to put a "Loewner collar" U around the smallest complementary component D_n with thickness proportional to the diameter of D_n with a proportionality coefficient depending on s and k, and is ϕ-Loewner with $\phi = \phi(s,k)$.

Proposition (Main statement)**.** *Let* $\Omega \in \mathbb{C}_\infty$*,*

$$\Omega = \mathbb{C}_\infty \backslash \bigcup_{i=1}^{n} D_i$$

where complementary components D_i *are* s*-relatively separated closed Jordan regions with boundaries* ∂D_i *being* k*-quasicircles. Then* Ω *is a* ϕ*-Loewner with* $\phi = \phi(n,s,k)$*.*

Proof. By induction on n. Case $n = 1$ is covered by **3.** For $n \geq 2$, consider arbitrary continua E and F in $\overline{\Omega}$ with $\Delta(E,F) \leq t$. We need to show that $\mathrm{Mod}(\Gamma(E,F;\Omega))$ is large, i.e. if a measure ρ has small mass $\int_\Omega \rho^2 d\Sigma < m$, then it is not admissible, i.e. exists a rectifiable path γ connecting E and F with $\int_\gamma \rho ds < 1$.

By induction hypothesis, such path already exists in the region with $n-1$ complementary components. Main idea: change the path such that it will not intersect the last component but will be still "short". WLOG assume D_n has the smallest diameter $d := \mathrm{diam}(D_n)$. There exists $m_1 = m_1(n,s,k,t) > 0$ such that if $\int \rho^2 d\Sigma < m_1$, then there exists a path α in $\Omega \cup D_n$ connecting E and F with

$$\int_\alpha \rho ds < 1/2.$$

We can assume that α intersects D_n. (Otherwise the claim is obvious.)

We will remove the piece of α in D_n and connect the remaining pieces by a path β in U, where U is a "Loewner collar" around D_n as in Collar Lemma with $\int_\beta \rho ds < 1/2$ to obtain the desired path γ in Ω. Let $c = c(s,k) > 0$ be a "thickness" constant from Collar Lemma, i.e. that $N_{cd}(D_n) \backslash D_n \subseteq U$. We will need to consider 3 cases:

Case 1. Neither E nor F is contained in $N_{\frac{1}{3}cd}(D_n)$.

Choose a closed (possibly degenerate) subpath α' of α from its endpoint x on E to the first point in $\overline{N_{\frac{1}{6}cd}(D_n)}$, call it x'. Since a continuum $\alpha' \cup E$ is not contained in $N_{\frac{1}{3}cd}(D_n)$, $\alpha' \cup E \backslash B(x',r) \neq \emptyset$, where $r := \frac{1}{6}cd$. We can find a continuum $E' \subseteq \alpha' \cup E$ that is contained in $\overline{B(x',r)}$ with $\mathrm{diam}\, E' \geq r = \frac{1}{6}cd$ (pick a connected component of $\alpha' \cup E \cap B(x',r)$ that contains x'). Then $E' \in \overline{U}$. Likewise, we can select a subpath α'' of α with endpoint on F and a subcontinuum F' of F in $\overline{U}$ with $\mathrm{diam}\, F' \geq \frac{1}{6}cd$. Then

$$\mathrm{dist}(E',F') \leq (2c+1)d \leq (12 + 6/c)\min(\mathrm{diam}(E'), \mathrm{diam}(F')),$$

and thus $\Delta(E', F') \leq C(s,k)$. Since U is ϕ-Loewner with $\phi = \phi(s,k)$, there exists $m_2 = m_2(s,k) > 0$ that if $\int \rho^2 d\Sigma < m_2$, then there exists a path β connecting E' and F' in U with $\int_\beta \rho ds < 1/2$.

Case 2. $t \min(\operatorname{diam}(E), \operatorname{diam}(F)) \geq \frac{1}{3}cd$.

Choose α' and α'' as in Case 1. Similarly, we can find continua $E' \subseteq \alpha' \cup E$ and $F' \subseteq \alpha'' \cup F$ with $E', F' \subseteq \overline{U}$ such that $\operatorname{diam}(E')) \geq \frac{1}{3}\min(\operatorname{diam}(E), cd)$ and $\operatorname{diam}(F')) \geq \frac{1}{3}\min(\operatorname{diam}(F), cd)$. Then

$$\operatorname{dist}(E', F') \leq (2c+1)d,$$

$$\min(\operatorname{diam}(E'), \operatorname{diam}(F')) \geq \frac{1}{3}\min(\operatorname{diam}(E), \operatorname{diam}(F), cd) \geq \frac{cd}{9\max(t,1)},$$

so $\Delta(E', F') \leq C(s,k,t)$. By Loewner property of U if $\int \rho^2 d\Sigma < m_3$, where $m_3 = m_3(s,k,t) > 0$, then there is a path β connecting E' and F' in U with $\int_\beta \rho ds < 1/2$.

Case 3. $t \min(\operatorname{diam}(E), \operatorname{diam}(F)) < \frac{1}{3}cd$ and either E or F lies in $N_{\frac{1}{3}cd}(D_n)$.

WLOG assume $E \in N_{\frac{1}{3}cd}(D_n)$. Then $E \subseteq \overline{U}$ and by the choice of t

$$\operatorname{dist}(E, F) \leq t \min(\operatorname{diam}(E), \operatorname{diam}(F)) < \frac{1}{3}cd.$$

Pick $x \in E$ and $y \in F$ with $\sigma(x,y) = \operatorname{dist}(E,F)$, and let $r = \frac{1}{3}\min(\operatorname{diam}(F), cd)$. There exists (similarly to the Case 1) a continuum $F' \subseteq F \cap \overline{B(y,r)}$ with $y \in F'$ and $\operatorname{diam}(F') \geq r = \frac{1}{3}\min(\operatorname{diam}(F), cd)$. Then

$$\operatorname{dist}(E, F') = \operatorname{dist}(E, F) < \frac{1}{3}cd,$$

$$\operatorname{dist}(E, F') \leq \min(t \operatorname{diam}(E), t \operatorname{diam}(F), \frac{1}{3}cd) \leq 3\max(t,1)\min(\operatorname{diam}(E), \operatorname{diam}(F')).$$

Thus $\Delta(E, F') \leq 3\max(t,1)$. Since U is Loewner, if $\int \rho^2 d\Sigma < m_4$, where $m_4 = m_4(s,k,t) > 0$, then there is a path β connecting E and F' in U with $\int_\beta \rho ds < 1$. Here we can simply take $\gamma = \beta$, since β connects E to $F' \subseteq F$.

Finally, we can take $m = \min\{m_1, m_2, m_3, m_4\} = m(n,s,k,t)$ and if $\int \rho^2 d\Sigma < m$, then we can find a path γ in Ω connecting E and F that satisfies $\int_\gamma \rho ds < 1$. $\square$