Backward iteration in the unit ball

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One-dimensional case
Forward iteration

Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$
n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n \ times}$

By **Schwarz's lemma**, f is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|$$

Theorem (Denjoy-Wolff)

If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.
if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$
if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

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The point p is called the **Denjoy-Wolff point** of f.

Cases:

1.$p \in \mathbb{D}$ f is called elliptic

2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic

3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

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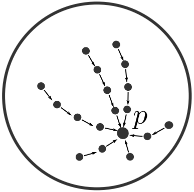

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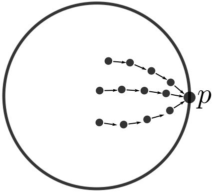

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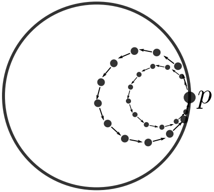

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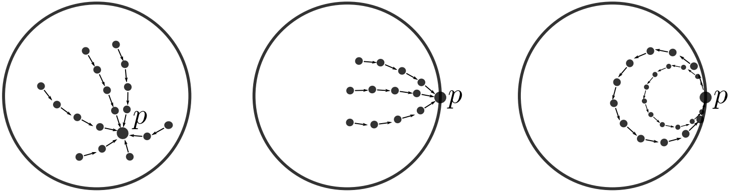

Elliptic Hyperbolic Parabolic

If $p \in \partial\mathbb{D}$, **Julia's lemma** holds for the point p, and multiplier $c = f'(p) \leq 1$:

$$\forall R > 0 \quad f\left(H(p, R)\right) \subseteq H(p, cR),$$

where $H(p, R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R :

$$H(p, R) := \left\{ z \in \mathbb{D} : \frac{|p - z|^2}{1 - |z|^2} < R \right\}$$

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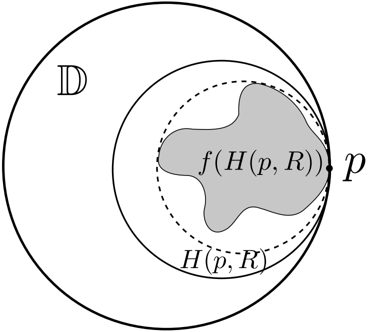

Backward iteration

Backward-iteration sequence: $\{z_n\}_{n=0}^{\infty}$, $f(z_{n+1}) = z_n$

The sequence $d(z_n, z_{n+1})$ is increasing, so we need a bound on the pseudo-hyperbolic step: $d(z_n, z_{n+1}) \leq a < 1$

Theorem (Poggi-Corradini, 2003)

Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map of the disk f with bounded pseudo-hyperbolic step $d(z_n, z_{n+1}) \leq a < 1$. Then:

1. $z_n \to q \in \partial\mathbb{D}$, and q is a fixed point with a well-defined multiplier $f'(q) < \infty$

2. If $q \neq p$, then q is a **boundary repelling fixed point** *(BRFP) (i.e. $f(q) = q$ and $1 < f'(q) < \infty$). If $q = p$, f is of parabolic type.*

3. When q is BRFP, the convergence $z_n \to q$ is non-tangential.

4. If $q = p$, then $z_n \to q$ tangentially.

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Multi-dimensional case

$\mathbb{C}^N$, inner product $(Z, W) = \sum_{j=1}^{N} Z_j \overline{W_j}$, $\|Z\|^2 = (Z, Z)$

Unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$

Julia's lemma in $\mathbb{B}^N$

Let f be a holomorphic self-map of $\mathbb{B}^N$ and $X \in \partial\mathbb{B}^N$ such that

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that $\forall R > 0$
$f(H(X, R)) \subset H(Y, \alpha R)$.

Horosphere of center $X \in \partial\mathbb{B}^N$ and radius $R > 0$:

$$H(X, R) = \left\{ Z \in \mathbb{B}^N : \frac{|1 - (Z, X)|^2}{1 - \|Z\|^2} < R \right\}$$

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Multi-dimensional version of Denjoy-Wolff theorem holds:

Theorem (MacCluer, 1983)

If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{Z\to p} \frac{1-\|f(Z)\|}{1-\|Z\|} \in (0,1]$ is a multiplier of f at p.
f is called **hyperbolic** *if $c < 1$ and* **parabolic** *if $c = 1$.*

We will call f **elliptic** if it has unique fixed point inside of the ball (WLOG fixed point is 0) and f is not unitary of any slice (i.e. with $\|f(Z)\| < \|Z\| \ \forall Z \in \mathbb{B}^N \setminus \{0\}$).

Siegel domain:
$\mathbb{H}^N = \{(z,w) \in \mathbb{C} \times \mathbb{C}^{N-1} : Rez > \|w\|^2\}$

Cayley transform: $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$
$\mathcal{C}((z,w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right) \quad \mathcal{C}^{-1}((z,w)) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right)$

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Theorem 1.(O —, 2010)

Let f be a analytic self-map of $\mathbb{B}^N$ *of hyperbolic or elliptic type,* $\{Z_n\}$ *be a backward-iteration sequence with bounded pseudo-hyperbolic step* $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$*. Then:*

1. *There exists a point* $\partial\mathbb{B}^N \ni \tau \neq p$ *such that* $Z_n \xrightarrow[n\to\infty]{} \tau$
2. $\{Z_n\}$ *stays in a Koranyi region with vertex* τ
3. *Julia's lemma holds for* τ *with multiplier* $\alpha \geq \frac{1}{c}$*, i.e.* $f(H(\tau, R)) \subset H(\tau, \alpha R)\ \forall R > 0$

Definition

A point $\tau \in \partial\mathbb{B}^N$ *is called a* **boundary repelling fixed point** *if Julia's lemma holds for* τ *with multiplier* $\alpha > 1$.

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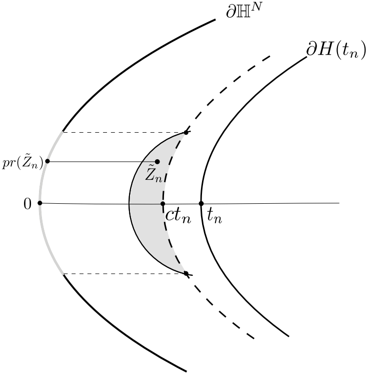

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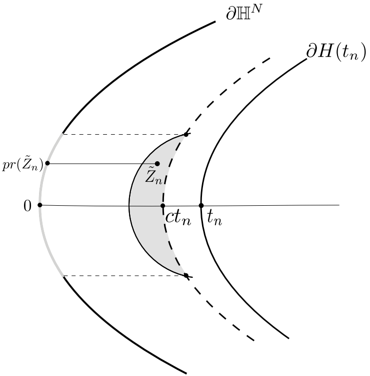

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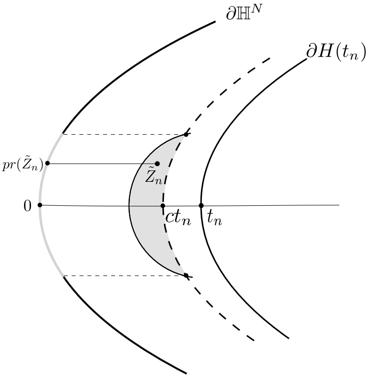

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In elliptic case we need the following

Lemma

Let f be a self-map of the unit ball $\mathbb{B}^N$ fixing zero, not unitary on any slice. Fix $r_0 > 0$, define $M(r) := \max \|f(r\mathbb{B}^N)\|$, $r \in [r_0, 1)$. Then there exists $c < 1$ such that

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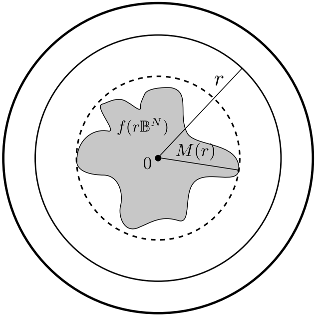

A BRFP with multiplier α is called **isolated** if it has a neighborhood with no other BRFPs with multiplier $\leq \alpha$.

In the hyperbolic and elliptic cases we have the following

Characterization of BRFP in terms of backward-iteration sequences:

Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

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Unlike in 1-dimensional case, not all BRFP's are isolated

Example 1. (O —, 2010):

$f : \mathbb{H}^2 \to \mathbb{H}^2$, $f(z, w) = (2z + w^2, w)$, *hyperbolic with multiplier* $1/2$ *at the Denjoy-Wolff point* ∞

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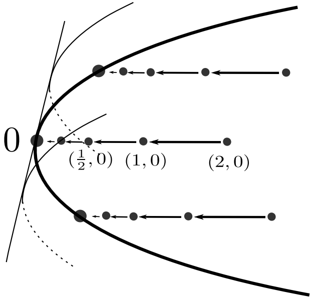

Definition

We will call the union of all backward iteration sequences with bounded step tending to a BRFP q a **stable set** *at q.*

The stable set at each BRFP (r, ir^2) in the Example 1 is $\{(z, r) \,|\, \mathrm{Re}\, z > r^2\}$ and has dimension 1.

Conjecture

BRFPs in $\mathbb{H}^N$ with stable set of dimension N are isolated.

(The conjecture is true for $N = 1$ since all BRFPs are isolated in 1-dimensional case).

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Suppose $f : \mathbb{H}^N \to \mathbb{H}^N$ is an analytic function and 0 is an isolated boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$. Then f is conjugated to the automorphism $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map ψ.

Construction of ψ:

$$\psi = \lim_{n \to \infty} \{f_n \circ \tau_n \circ p_1\}$$

where $p_1(z, w) := (z, 0)$ is the projection on the first (radial) dimension, so

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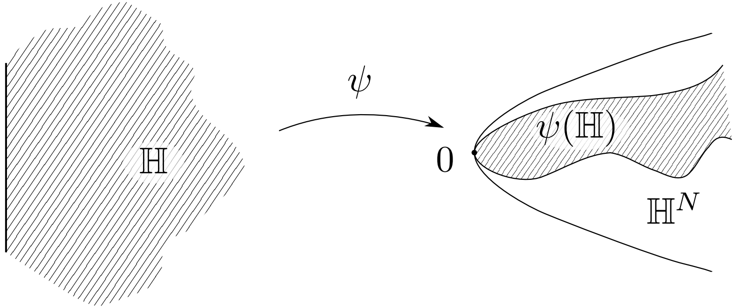

Theorem 3. (O —, 2009)

Under some regularity condition, it is possible to improve ψ such that

$$\psi(z,w) = \psi(p_L(z,w)),$$

where p_L is a projection on the first L dimensions.

Condition is

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Subcases (do not depend on the choice of sequence):

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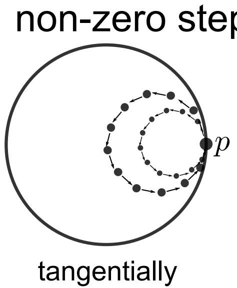

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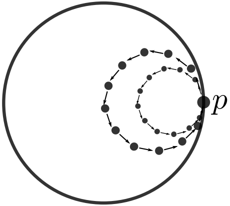
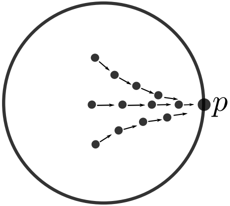


tangentially

zero step



radially

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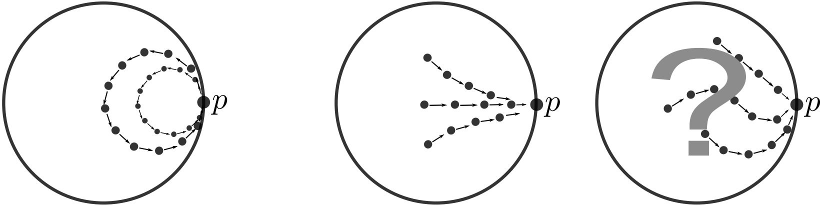

tangentially　　　　radially　　　　other: not known

Parabolic case in the ball: Zero and non-zero step cases are defined only for sequences.

Claim

If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ *for parabolic self-map of the unit ball is restricted, then it must have zero step, i.e.* $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$. *In particular, non-zero-step sequence cannot converge non-tangentially.*

The only known parabolic examples in $\mathbb{H}^N$ are:

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$(z, w) \longmapsto (z + z_0 + 2\langle w, w_0\rangle, w + w_0)$ *for some* $(z_0, w_0) \in \partial\mathbb{H}^N$.

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Given one-dimensional $\phi : \mathbb{H} \to \mathbb{H}$ *of hyperbolic or parabolic type, with the Denjoy-Wolff point* ∞ *and BRFP* iy_0*,*
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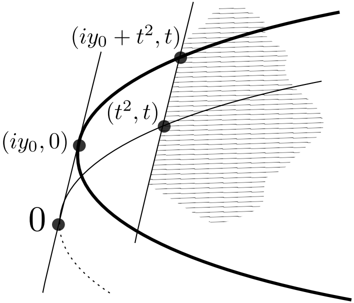

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