Backward iteration in the unit ball

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Analysis/PDE Seminar
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Outline of my Talk

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1. One-dimensional case
   - Forward iteration
   - Backward iteration
2. Multi-dimensional case
   - Preliminaries
   - Main result and examples
3. Conjugations
   - Overview
   - Conjugations near BRFP in the unit ball
4. Parabolic case
5. Future goals

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One-dimensional case
Forward iteration

Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$
n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n \text{ times}}$

By **Schwarz's lemma**, f is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|$$

Theorem (Denjoy-Wolff)

If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.
if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$
if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

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The point p is called the **Denjoy-Wolff point** of f.

Cases:

1.$p \in \mathbb{D}$ f is called elliptic

2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic

3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

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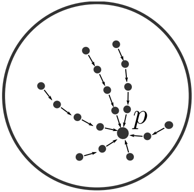

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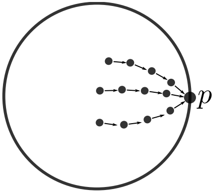

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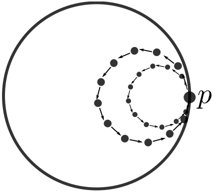

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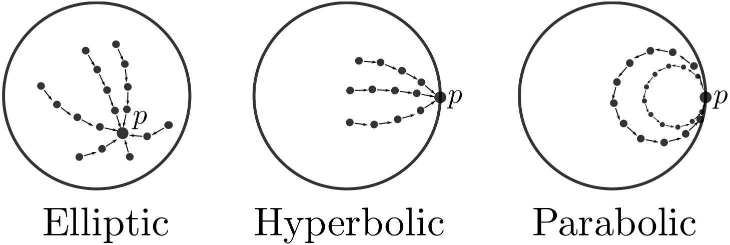

If $p \in \partial\mathbb{D}$, **Julia's lemma** holds for the point p, and multiplier $c = f'(p) \leq 1$:

$$\forall R > 0 \quad f(H(p,R)) \subseteq H(p,cR),$$

where $H(p,R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R :

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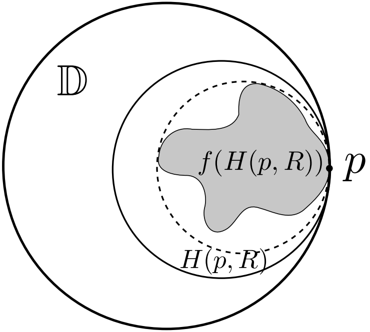

Backward iteration

Backward-iteration sequence: $\{z_n\}_{n=0}^{\infty}$, $f(z_{n+1}) = z_n$

Not always exists: $f(z) = cz$, $|c| < 1$ has no backward iteration sequences.

By Schwarz's lemma, $d(z_{n+1}, z_n) \geq d(z_n, z_{n-1})$ $\forall n$, so $d_n := d(z_{n+1}, z_n) \nearrow$.

We need additional condition on sequence to converge:

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(the pseudo-hyperbolic step must be bounded above).

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Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map (not an elliptic automorphism) of the disk f with bounded pseudo-hyperbolic step $d(z_n, z_{n+1}) \leq a < 1$. Then:

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- *The sequence converges to the point on the boundary $q \in \partial\mathbb{D}$, and q is a fixed point with a well-defined derivative $f'(q) < \infty$*
- *If $q \neq p$, then q is a* **boundary repelling fixed point** *(BRFP) (i.e. $f'(q) > 1$). The convergence $z_n \to q$ is non-tangential.*

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- *If* $q \neq p$*, then* q *is a* **boundary repelling fixed point** *(BRFP) (i.e.* $f'(q) > 1$*). The convergence* $z_n \to q$ *is non-tangential.*
- *If* $q = p$*, then* $z_n \to q$ *tangentially. It may happen only in parabolic case.*

Multi-dimensional case

$\mathbb{C}^N$, inner product $(Z, W) = \sum_{j=1}^{N} Z_j \overline{W_j}$, $\|Z\|^2 = (Z, Z)$

Unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$

Julia's lemma in $\mathbb{B}^N$

Let f be a holomorphic self-map of $\mathbb{B}^N$ and $X \in \partial\mathbb{B}^N$ such that

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that $\forall R > 0$
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Horosphere of center $X \in \partial\mathbb{B}^N$ and radius $R > 0$:

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Multi-dimensional version of Denjoy-Wolff theorem holds:

Theorem (MacCluer, 1983)

If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{Z \to p} \frac{1-\|f(Z)\|}{1-\|Z\|} \in (0,1]$ is a multiplier of f at p.
f is called **hyperbolic** *if $c < 1$ and* **parabolic** *if $c = 1$.*

We will call f **elliptic** if it has unique fixed point inside of the ball (WLOG fixed point is 0) and f is not unitary of any slice (i.e. with $\|f(Z)\| < \|Z\| \; \forall Z \in \mathbb{B}^N \setminus \{0\}$).

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Siegel domain: $\mathbb{H}^N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : Rez > \|w\|^2\}$

is biholomorphically equivalent to the unit ball $\mathbb{B}^N$ via **Cayley transform:** $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$$\mathcal{C}((z, w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right) \quad \mathcal{C}^{-1}((z, w)) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right)$$

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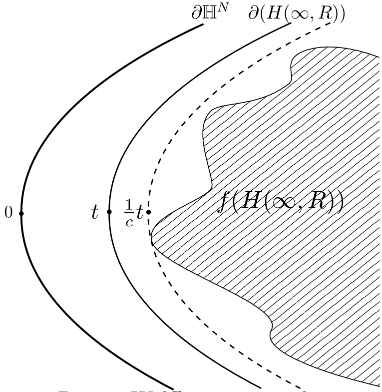

Horosphere at Denjoy-Wolff point ∞ and its image in $\mathbb{H}^N$

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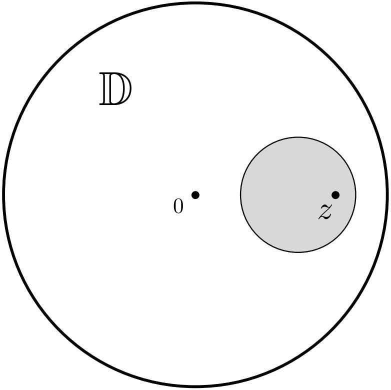
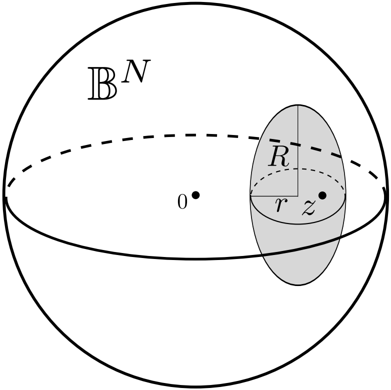


Pseudo-hyperbolic disk
is a Euclidean disk



Pseudohyperbolic ball is
a Euclidean ellipsoid with $R > r$
and $\frac{R}{r} \to \infty$ as $z \to \partial\mathbb{B}^N$

Theorem 1.(O —, 2010)

Let f be a analytic self-map of $\mathbb{B}^N$ of hyperbolic or elliptic type, $\{Z_n\}$ be a backward-iteration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:

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2. $\{Z_n\}$ *stays in a Koranyi region with vertex q (Koranyi regions are weaker analogs of non-tangential regions in higher dimension).*

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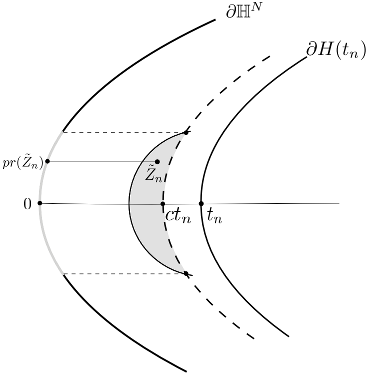
Definition

A point $q \in \partial\mathbb{B}^N$ is called a **boundary repelling fixed point** *if Julia's lemma holds for q with multiplier $\alpha > 1$.*

Idea of the proof in hyperbolic case:



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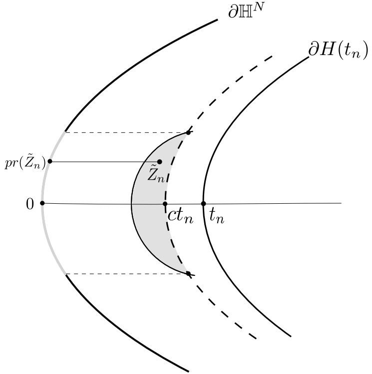

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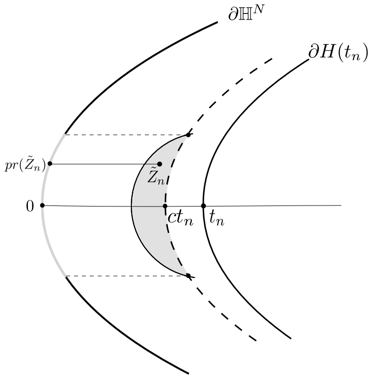

$t_n := \operatorname{Re} z_n - \|w_n\|^2 \sim c^n$ (by Julia's lemma)

$\|pr(Z_n) - pr(Z_{n+1})\| \leq C\sqrt{t_n} \sim c^{n/2}$

In elliptic case we need the following

Lemma

Let f be a self-map of the unit ball $\mathbb{B}^N$ *fixing zero, not unitary on any slice. Fix* $r_0 > 0$*, define* $M(r) := \max \|f(r\mathbb{B}^N)\|$*,* $r \in [r_0, 1)$*. Then there exists* $c < 1$ *such that*

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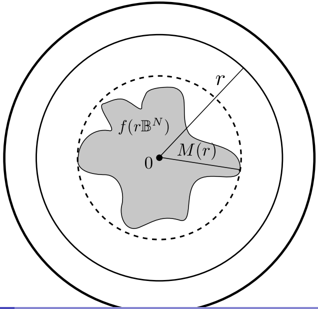

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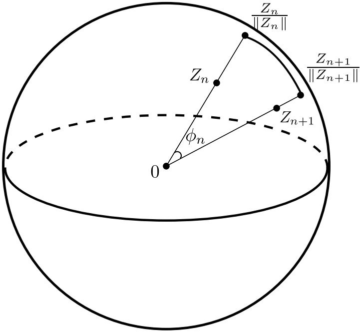

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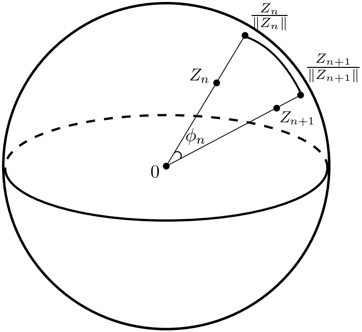

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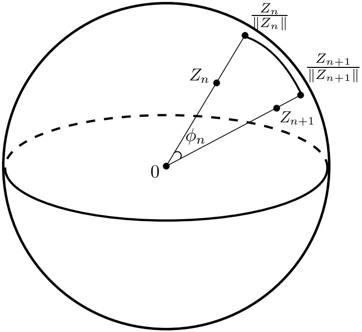

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A BRFP with multiplier α is called **isolated** if it has a neighborhood with no other BRFPs with multiplier $\leq \alpha$.

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Characterization of BRFP in terms of backward-iteration sequences:

Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

In 1-dimensional case all boundary fixed points are isolated (corollary of the theorem of Cowen and Pommerenke, 1982), so the above characterization is "if and only if".

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$f : \mathbb{H}^2 \to \mathbb{H}^2$, $f(z, w) = (2z + w^2, w)$, *hyperbolic with multiplier* $1/2$ *at the Denjoy-Wolff point* ∞

Iterates: $f_n(z, w) = (2^n z + (2^n - 1)w^2, w)$
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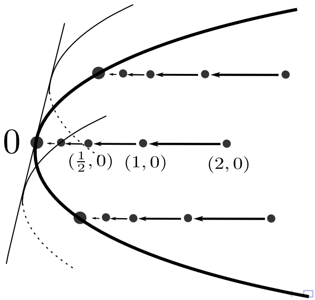

Definition

We will call the union of all backward iteration sequences with bounded step tending to a BRFP q a **stable set** *at q.*

The stable set at each BRFP (r, ir^2) in the Example 1 is $\{(z, r) \,|\, \mathrm{Re}\, z > r^2\}$ and has dimension 1.

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For self-map f of $\mathbb{D}$ *(or* $\mathbb{B}^N$*), solve an equation*

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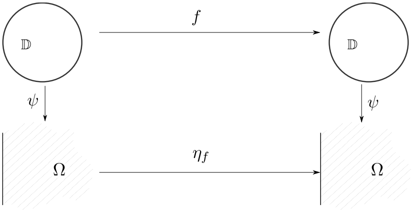

Koenigs, 1884

If f *is* **elliptic** *with* $f'(p) \neq 0$, *then*

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Valiron, 1913

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Pommerenke, Baker and Pommerenke, 1979

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Poggi-Corradini, 2000 (backward iteration):

An analytic self-map of the unit disc $\mathbb{D}$ f with BRFP $1 \in \partial\mathbb{D}$ and multiplier α at 1 can be conjugated to the automorphism $\eta(z) = (z - a)/(1 - az)$, where $a = (\alpha - 1)/(\alpha + 1)$:

$$\psi \circ \eta(z) = f \circ \psi(z),$$

via an analytic map ψ of $\mathbb{D}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$, which has non-tangential limit 1 at 1.

Conjugations in several dimensions

Bracci, Gentili, Poggi-Corradini, 2010; hyperbolic case

Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be a hyperbolic analytic self-map with Denjoy-Wolff point $p \in \partial\mathbb{B}^N$ and multiplier $c < 1$. If

1. *There exists special sequence $f_n(Z_0) \to p$ and*
2. *the $K - \lim\limits_{Z \to p} \dfrac{1 - \langle f(Z), p\rangle}{1 - \langle Z, p\rangle}$ exists,*

then there is a non-constant analytic function $\psi : \mathbb{B}^N \to \mathbb{H}$ such that

$$\psi \circ f = \frac{1}{c} \cdot \psi$$

Theorem 2. (O —, 2009) (N-dimensional case, backward iteration)

Suppose $f : \mathbb{H}^N \to \mathbb{H}^N$ is an analytic function and 0 is an isolated boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$. Then f is conjugated to the automorphism $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

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Construction of ψ:

$$\psi = \lim_{n \to \infty} \{f_n \circ \tau_n \circ p_1\}$$

where $p_1(z, w) := (z, 0)$ is the projection on the first (radial) dimension, so

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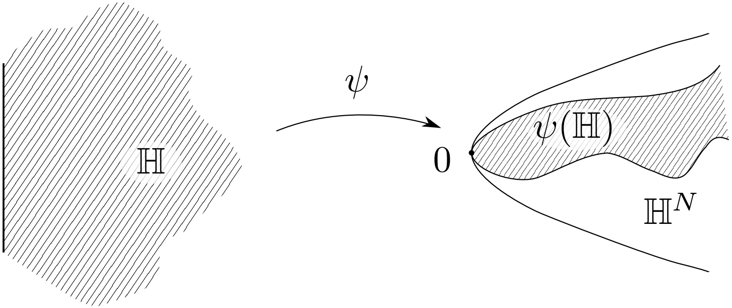

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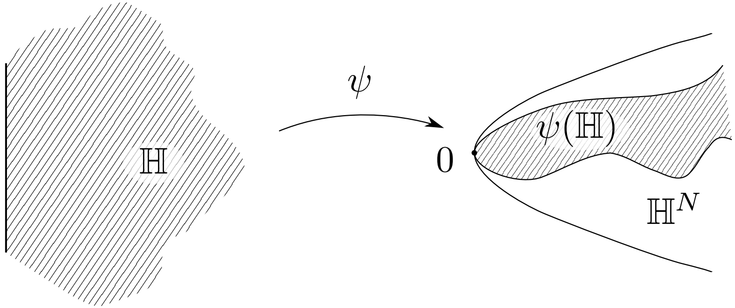

Corollary

Since image of ψ is always a subset of stable set, the dimension of stable set is at least 1.

Theorem 3. (O —, 2009)

Under some regularity condition, it is possible to improve ψ such that

$$\psi(z,w) = \psi(p_L(z,w)),$$

where p_L is a projection on the first L dimensions.

Condition is

$$f(z,w) = (\alpha z + o(|z|), Aw + o(|z|^{1/2}))$$

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Parabolic case in the disk

Since $d(z_n, z_{n+1}) \leq d(z_{n-1}, z_n)$, pseudo-hyperbolic step $d_n := d(z_n, z_{n+1})$ must have limit: $d_n \xrightarrow[n\to\infty]{} b$

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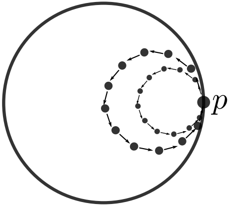

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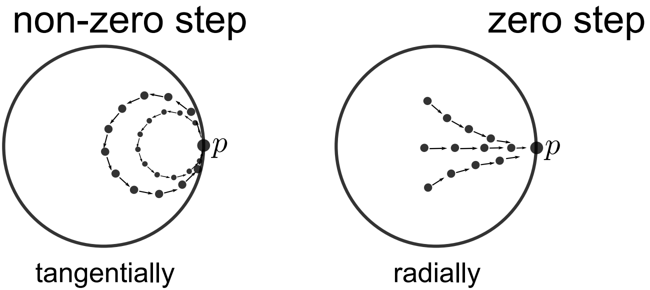

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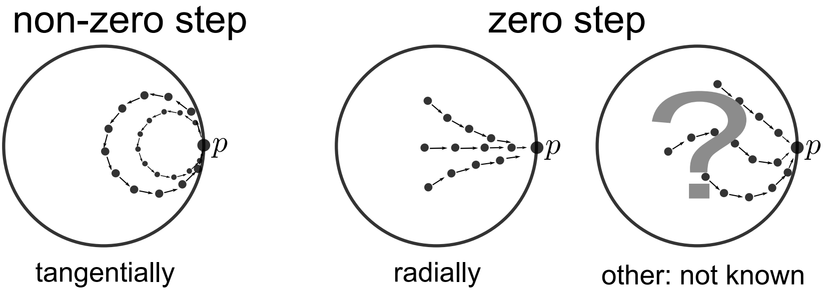

Parabolic case in the ball: Zero and non-zero step cases are defined only for sequences.

Open question

Is it true that if $d_{\mathbb{B}^N}(f_n(Z_0), f_{n+1}(Z_0)) \to 0$ for some $Z_0 \in \mathbb{B}^N$, then $d_{\mathbb{B}^N}(f_n(Z), f_{n+1}(Z)) \to 0$ for all $Z \in \mathbb{B}^N$?

Claim

If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ for parabolic self-map of the unit ball is restricted, then it must have zero step, i.e. $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$. In particular, non-zero-step sequence cannot converge non-tangentially.

The only known parabolic examples in $\mathbb{H}^N$ are:

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$(z, w) \longmapsto (z + z_0 + 2\langle w, w_0\rangle, w + w_0)$ *for some* $(z_0, w_0) \in \partial\mathbb{H}^N$.

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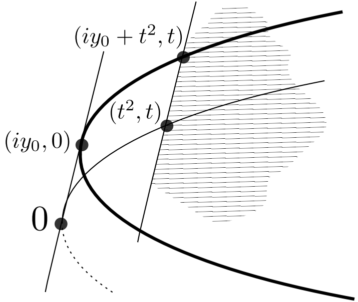

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Thank you!

http://arxiv.org/abs/0910.5451