Backward iteration in the unit ball

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Conference on Complex Analysis
in honor of David Drasin and Linda Sons
University of Illinois at Urbana-Champaign

One-dimensional case
Forward iteration

Let f be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$
n-th iterate of f $f_n = \underbrace{f \circ \ldots \circ f}_{n \ times}$

By **Schwarz's lemma**, f is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \overline{w} z} \right|$$

Theorem (Denjoy-Wolff)

If a self-map of the disk f is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to p.
if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$
if $p \in \partial\mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

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The point p is called the **Denjoy-Wolff point** of f.

Cases:

1.$p \in \mathbb{D}$ f is called elliptic

2.$p \in \partial\mathbb{D}$, $f'(p) < 1$ hyperbolic

3.$p \in \partial\mathbb{D}$, $f'(p) = 1$ parabolic

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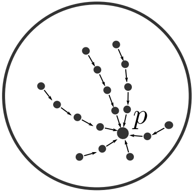

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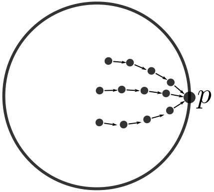

Hyperbolic

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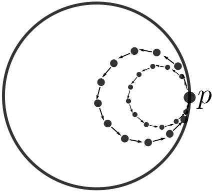

Parabolic

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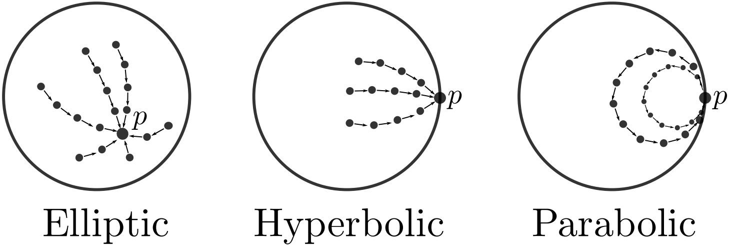

If $p \in \partial\mathbb{D}$, **Julia's lemma** holds for the point p, and multiplier $c = f'(p) \leq 1$:

$$\forall R > 0 \quad f\left(H(p,R)\right) \subseteq H(p,cR),$$

where $H(p,R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R :

$$H(p,R) := \left\{ z \in \mathbb{D} : \frac{|p-z|^2}{1-|z|^2} < R \right\}$$

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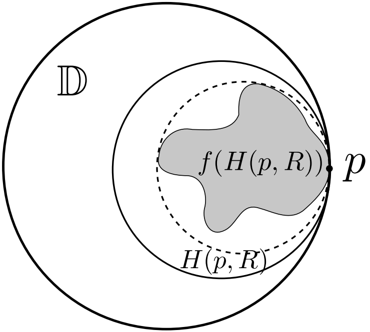

Backward iteration

Backward-iteration sequence:
$\{z_n\}_{n=0}^{\infty}$, $f(z_{n+1}) = z_n$ for $n = 0, 1, 2 \ldots$
The sequence $d(z_n, z_{n+1})$ is increasing, so we need a bound on the pseudo-hyperbolic step: $d(z_n, z_{n+1}) \leq a < 1$

Theorem (Poggi-Corradini, 2003)

Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map of the disk f with bounded pseudo-hyperbolic step $d(z_n, z_{n+1}) \leq a < 1$. Then:
1. $z_n \to q \in \partial\mathbb{D}$, and q is a fixed point with a well-defined multiplier $f'(q) < \infty$
2. If $q \neq p$, then q is a **boundary repelling fixed point** *(BRFP) (i.e. $f'(q) > 1$). If $q = p$, f is of parabolic type.*
3. When q is BRFP, the convergence $z_n \to q$ is non-tangential.
4. If $q = p$, then $z_n \to q$ tangentially.

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Multi-dimensional case

$\mathbb{C}^N$, inner product $(Z, W) = \sum_{j=1}^{N} Z_j \overline{W_j}$, $\|Z\|^2 = (Z, Z)$

Unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$

Julia's lemma in $\mathbb{B}^N$

Let f be a holomorphic self-map of $\mathbb{B}^N$ and $X \in \partial\mathbb{B}^N$ such that

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that $\forall R > 0$

$f(H(X, R)) \subset H(Y, \alpha R)$.

Horosphere of center $X \in \partial\mathbb{B}^N$ and radius $R > 0$:

$$H(X, R) = \left\{ Z \in \mathbb{B}^N : \frac{|1 - (Z, X)|^2}{1 - \|Z\|^2} < R \right\}$$

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Multi-dimensional version of Denjoy-Wolff theorem holds:

Theorem (MacCluer, 1983)

If f has no fixed points in $\mathbb{B}^N$, then f_n converges uniformly on compacta to $p \in \partial\mathbb{B}^N$, the number $c := \liminf_{Z\to p} \frac{1-\|f(Z)\|}{1-\|Z\|} \in (0,1]$ is a multiplier of f at p.

f is called hyperbolic if $c < 1$ and parabolic if $c = 1$.

Siegel domain:

$\mathbb{H}^N = \{(z,w) \in \mathbb{C} \times \mathbb{C}^{N-1} : Rez > \|w\|^2\}$

Cayley transform: $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$$\mathcal{C}((z,w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right)$$

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Elliptic case: f has unique fixed point inside of the ball (WLOG fixed point is 0) and f is not unitary of any slice (i.e. with $\|f(Z)\| < \|Z\| \; \forall Z \in \mathbb{B}^N \backslash \{0\}$).

Theorem 1.

Let f be a analytic self-map of $\mathbb{B}^N$ of hyperbolic or elliptic type, $\{Z_n\}$ be a backward-iteration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:

1. There exists a point $\partial\mathbb{B}^N \ni \tau \neq p$ such that $Z_n \xrightarrow[n\to\infty]{} \tau$

2. $\{Z_n\}$ stays in a Koranyi region with vertex τ

3. Julia's lemma holds for τ with multiplier $\alpha \geq \frac{1}{c}$, i.e. $f(H(\tau, R)) \subset H(\tau, \alpha R) \; \forall R > 0$

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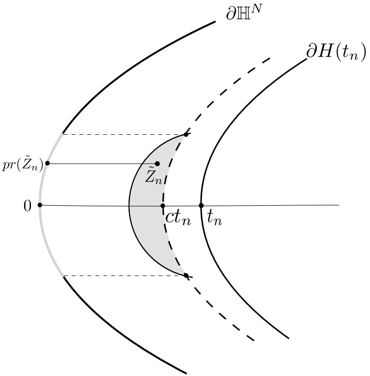
2. $\{Z_n\}$ stays in a Koranyi region with vertex τ
3. Julia's lemma holds for τ with multiplier $\alpha \geq \frac{1}{c}$, i.e.
$f(H(\tau, R)) \subset H(\tau, \alpha R) \; \forall R > 0$

Idea of the proof in hyperbolic case:



$t_n := \mathrm{Re}\, z_n - \|w_n\|^2 \sim c^n$ (by Julia's lemma)

$\|pr(Z_n) - pr(Z_{n+1})\| \leq C\sqrt{t_n} \sim c^{n/2}$

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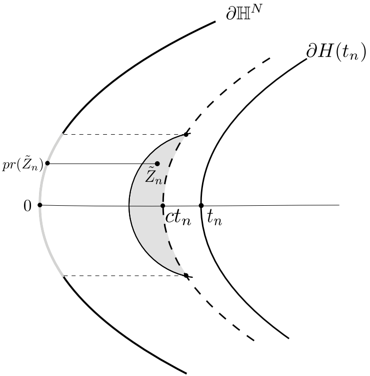

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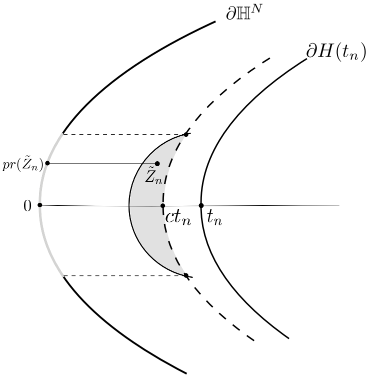

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Lemma

Let f be a self-map of the unit ball $\mathbb{B}^N$ *fixing zero, not unitary on any slice. Fix* $r_0 > 0$, *define* $M(r) := \max \|f(r\mathbb{B}^N)\|$, $r \in [r_0, 1)$. *Then there exists* $c < 1$ *such that*

$$\frac{1-r}{1-M(r)} \leq c \quad \forall r \in [r_0, 1)$$

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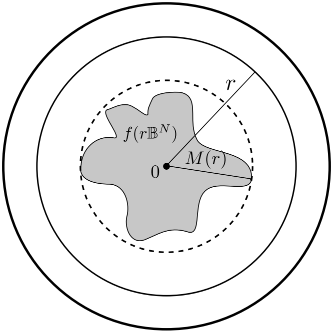
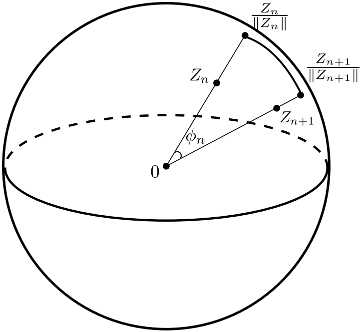


Idea of the proof in elliptic case:



$t_n := 1 - \|Z_n\| \sim c^n$ (by lemma)

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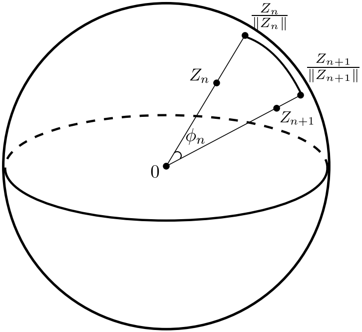

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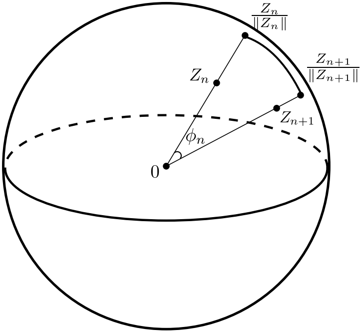

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Since $\alpha \geq \frac{1}{c} > 1$, the point $\tau \in \partial\mathbb{B}^N$ is called the **boundary repelling fixed point** (BRFP) for f.

Characterization of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

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Problem:

Unlike in 1-dimensional case, not all BRFP's are isolated

Counterexample: $f : \mathbb{H}^2 \to \mathbb{H}^2$, $f(z, w) = (2z + w^2, w)$

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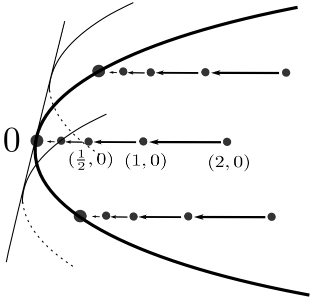

Conjugations

Theorem 2. (N-dimensional case, backward iteration)

Suppose $f : \mathbb{H}^N \to \mathbb{H}^N$ *is an analytic function and* 0 *is an isolated boundary repelling fixed point for* f *with multiplier* $1 < \alpha < \infty$. *Then* f *is conjugated to the automorphism* $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map ψ.

Construction of ψ:

$$\psi = \lim_{n \to \infty} \{f_n \circ \tau_n \circ p_1\}$$

where $p_1(z, w) := (z, 0)$ is the projection on the first (radial) dimension, so

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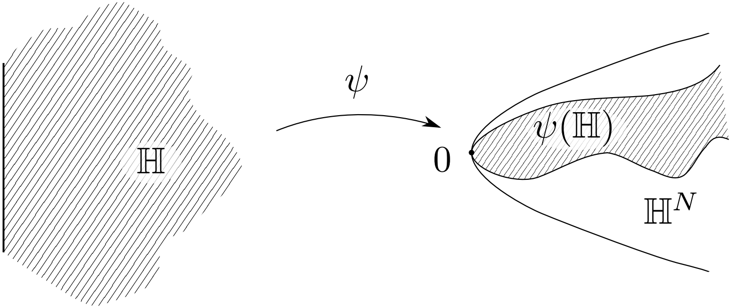

Conjugation for expandable maps

Theorem 3.

Under some regularity condition, it is possible to improve ψ such that

$$\psi(z, w) = \psi(p_L(z, w)),$$

where p_L is a projection on the first L dimensions.

Condition is

$$f(z, w) = (\alpha z + o(|z|), Aw + o(|z|^{1/2}))$$

e.g. $A = Diag\,(\sqrt{\alpha}, \ldots \sqrt{\alpha}, \beta_1, \ldots \beta_{N-L})$, where $\beta_j < \sqrt{\alpha}$

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Future goals

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• Conjugation for non-isolated fixed points

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Parabolic case in the disk

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Subcases (do not depend on the choice of sequence):

$b > 0$ parabolic non-zero step type

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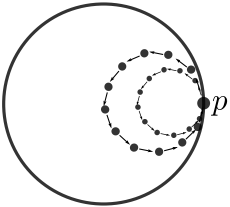

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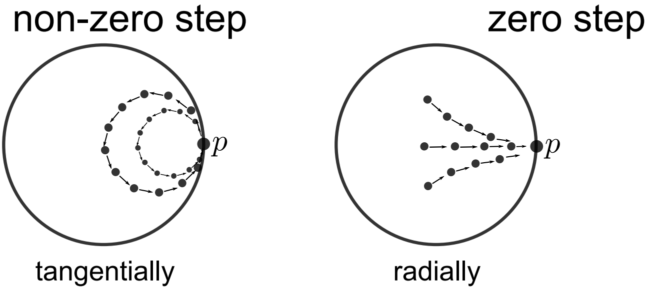

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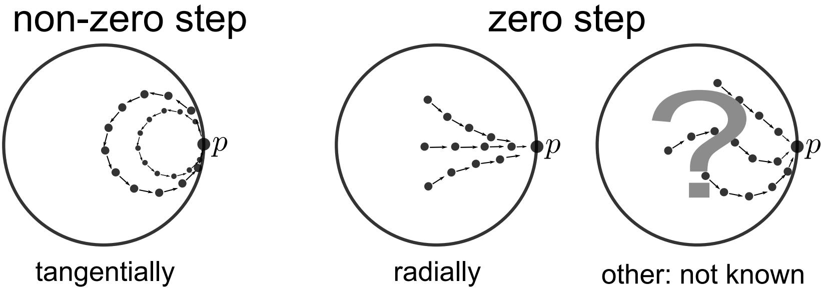

Parabolic case in the ball: Zero and non-zero step cases only for sequences.

Claim.

If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ *for parabolic self-map of the unit ball is restricted, then it must have zero step, i.e.* $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$. *In particular, non-zero-step sequence cannot converge non-tangentially.*

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