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RESEARCH STATEMENT

I study properties of backward iterates of analytic self-maps of the unit ball in $\mathbb{C}^N$. Many facts were established about such maps in the 1-dimensional case (i.e. for self-maps of the unit disk), and I focus on generalizing some of them in higher dimension.

1. One-dimensional case

1.1. **Forward iteration.** Let f be an analytic self-map of the unit disk $\mathbb{D}$. Denote $f_n = f^{\circ n}$ and consider the sequence of forward iterates $z_n = f_n(z_0)$. By Schwarz's lemma, f is a contraction of the pseudo-hyperbolic metric, so the sequence $d(z_n, z_{n+1})$ is decreasing, where

$$d(z,w) := \left| \frac{z-w}{1-\overline{w}z} \right|, \quad \forall z, w \in \mathbb{D}.$$

Theorem 1.1 (Denjoy-Wolff)**.** *If f is not an elliptic automorphism, then there exists a unique point $p \in \overline{\mathbb{D}}$ (called the Denjoy-Wolff point of f) such that the sequence of iterates $\{f_n\}$ converges to p uniformly on compact subsets of $\mathbb{D}$.*

Consider first the case $p \in \partial\mathbb{D}$. It can be shown that $f(p) = p$ and $f'(p) = c \leq 1$ in the sense of non-tangential limits, and the point p can thus be called "attracting". More geometrically, Julia's lemma holds for the point p, i.e.

$$\forall R > 0 \quad f\left(H(p,R)\right) \subseteq H(p, cR), \tag{1.1}$$

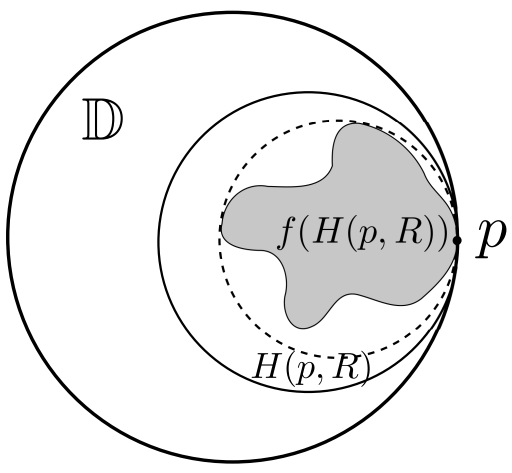

Figure 1. Julia's lemma at the Denjoy-Wolff point $p \in \partial\mathbb{D}$.

where $H(p,R)$ is a horocycle at $p \in \partial\mathbb{D}$ of radius R (see Figure 1),

$$H(p,R) := \left\{ z \in \mathbb{D} : \frac{|p-z|^2}{1-|z|^2} < R \right\}.$$

Here $c = f'(p)$ is the smallest c such that (1.1) holds. We will call it the multiplier or the dilatation coefficient and we will distinguish the hyperbolic ($c < 1$) and parabolic ($c = 1$) cases.

In the hyperbolic case, Valiron [16] showed that there is an analytic map $\psi : \mathbb{D} \to \mathbb{H}$ (where $\mathbb{H}$ is the right half-plane) with some regularity properties, which solves the Schröder equation:

$$\psi \circ f = \frac{1}{c}\psi, \tag{1.2}$$

and so ψ conjugates f to multiplication in $\mathbb{H}$.

In the parabolic case, f can be conjugated to a shift in a half-plane or in the whole plane, as proved by Pommerenke [15], and Baker and Pommerenke [2].

If the Denjoy-Wolff point p is in $\mathbb{D}$, the function f is said to be elliptic and the multiplier $c = f'(p)$ satisfies $|c| < 1$, unless f is an elliptic automorphism. Conjugations for such maps were found by Koenigs [10] and Böttcher [4].

Typical orbits for elliptic, hyperbolic and parabolic cases are shown in Figure 2.

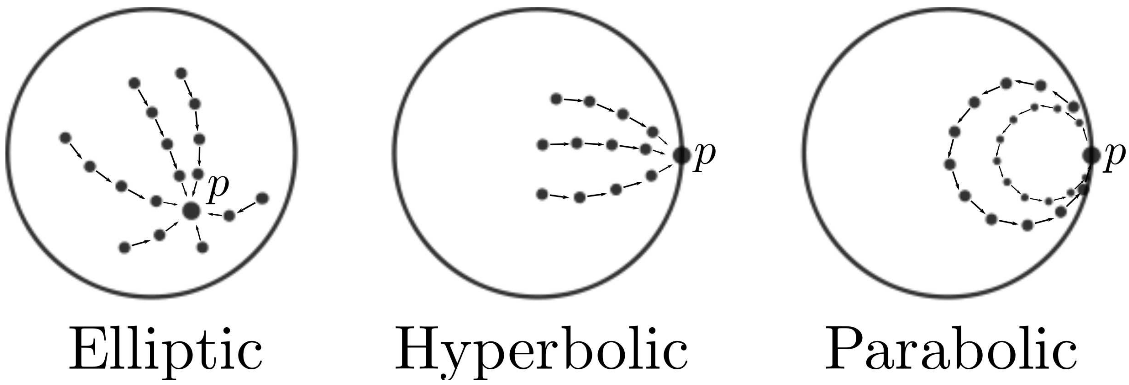

FIGURE 2. Denjoy-Wolff point p and typical orbits in elliptic, hyperbolic and parabolic cases.

1.2. **Backward iteration.**

Definition 1.2. We will call a sequence of points $\{z_n\}_{n=0}^{\infty}$ a backward-iteration sequence for f if $f(z_{n+1}) = z_n$ for $n = 0, 1, 2, \ldots$.

In general, such sequences may not exist. Note that in the backward iteration case the sequence $d(z_n, z_{n+1})$ is increasing, so we will impose an upper bound on the pseudo-hyperbolic step:

$$d(z_n, z_{n+1}) \le a, \quad \forall n, \tag{1.3}$$

for some fixed $a < 1$.

This condition is nontrivial, for an example of a map that admits a backward-iteration sequence with unbounded steps, see section 2 of [14].

A backward-iteration sequence satisfying (1.3) must converge to a point on the boundary of $\mathbb{D}$:

Theorem 1.3 (Poggi-Corradini, [12]). *Suppose f is an analytic map with $f(\mathbb{D}) \subseteq \mathbb{D}$ (and not an elliptic automorphism). Let $\{z_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for f with bounded pseudo-hyperbolic steps $d_n = d(z_n, z_{n+1}) \uparrow a < 1$. Then the following hold:*

(1) *There is a point $q \in \partial\mathbb{D}$ such that $z_n \to q$ as n tends to infinity, and q is a fixed point for f with a well-defined multiplier $f'(q) = \alpha < \infty$.*
(2) *When $q \neq p$, where p is the Denjoy-Wolff point, then $\alpha > 1$, so we can call q a* boundary repelling fixed point. *If $q = p$, then f is necessarily of parabolic type.*
(3) *When $q \neq p$, then the sequence z_n tends to q along a non-tangential direction.*
(4) *When, in the parabolic case, $q = p$, then z_n tends to q tangentially.*

In this case Julia's lemma holds for the point q with multiplier $\alpha > 1$:

$$\forall R > 0 \quad f\left(H(q, R)\right) \subseteq H(q, \alpha R), \tag{1.4}$$

where α is the smallest number such that this holds.

For backward iteration, the following conjugation result was obtained in [13]:

Theorem 1.4 (Poggi-Corradini). *Suppose f is an analytic self-map of the unit disc $\mathbb{D}$ and 1 is a boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$. Let $a = (\alpha - 1)/(\alpha + 1)$ and $\eta(z) = (z - a)/(1 - az)$. Then there is an analytic map ψ of $\mathbb{D}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$, which has non-tangential limit 1 at 1, such that*

$$\psi \circ \eta(z) = f \circ \psi(z), \tag{1.5}$$

for all $z \in \mathbb{D}$.

2. Unit ball in $\mathbb{C}^N$.

2.1. **Preliminaries.** Consider the N-dimensional unit ball $\mathbb{B}^N = \left\{Z \in \mathbb{C}^N : \|Z\| < 1\right\}$, where the inner product and the norm are defined as

$$(Z, W) = \sum_{j=1}^{N} Z_j \overline{W_j} \quad \text{and} \quad \|Z\|^2 = (Z, Z).$$

Schwarz's lemma still holds for a self-map f of the unit ball, i.e. f must be a contraction in the Bergmann metric $k_{\mathbb{B}^N}$ (Corollary (2.2.18) from [1]).

We also have the following generalization of Julia's lemma:

Theorem 2.1 (Theorem (2.2.21) from [1])**.** *Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be a holomorphic map and take $X \in \partial\mathbb{B}^N$ such that*

$$\liminf_{Z \to X} \frac{1 - \|f(Z)\|}{1 - \|Z\|} = \alpha < \infty. \tag{2.1}$$

Then there exists a unique $Y \in \partial\mathbb{B}^N$ such that

$$\forall R > 0 \quad f\left(H(X, R)\right) \subseteq H(Y, \alpha R),$$

where $H(X, R)$ is a horosphere (the N-dimensional generalization of a horocycle), defined as

$$H(X, R) := \left\{ Z \in \mathbb{B}^N : \frac{|1 - (Z, X)|^2}{1 - \|Z\|^2} < R \right\}.$$

And a version of the Denjoy-Wolff theorem also holds:

Theorem 2.2 (MacCluer, [11])**.** *Let $f : \mathbb{B}^N \to \mathbb{B}^N$ be a holomorphic map without fixed points in $\mathbb{B}^N$. Then the sequence of iterates $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{B}^N$ to the constant map $Z \mapsto p$ for a (unique) point $p \in \partial\mathbb{B}^N$ (called the Denjoy-Wolff point of f); and the number*

$$c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1] \tag{2.2}$$

is called the multiplier or the boundary dilatation coefficient of f at p.

The map f is called hyperbolic if $c < 1$ and parabolic if $c = 1$.

Unlike in the one-dimensional case, there may be many fixed points inside the unit ball $\mathbb{B}^N$. Even if the fixed point is unique, forward iterates need not converge to it (consider rotations). We will call a function f unitary on a slice if there exist ζ and η in $\partial\mathbb{B}^N$ with $f(\lambda\zeta) = \lambda\eta$ for all $\lambda \in \mathbb{D}$. Functions that are not unitary on any slice are precisely those for which strict inequality occurs in the multidimensional Schwarz lemma and for them forward iterates converge to 0 (see [7]).

Definition 2.3. We will call a self-map of the unit ball f elliptic, if it has a unique fixed point inside $\mathbb{B}^N$ and it is conjugate via an automorphism to a self-map fixing zero, which is not unitary on any slice.

I will consider only self-maps of the ball that are elliptic, hyperbolic or parabolic.

Sometimes it will be more convenient to use the Siegel domain:

$$\mathbb{H}^N := \left\{ (z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \operatorname{Re} z > \|w\|^2 \right\},$$

which is biholomorphic to $\mathbb{B}^N$ via the Cayley transform $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$:

$$\mathcal{C}(z, w) = \left(\frac{1 + z}{1 - z}, \frac{w}{1 - z} \right) \quad \text{and} \quad \mathcal{C}^{-1}(z, w) = \left(\frac{z - 1}{z + 1}, \frac{2w}{z + 1} \right).$$

I will use the same notations for the points in $\mathbb{B}^N$ and their images in $\mathbb{H}^N$, when this is not likely to cause confusion. I will also denote by (z, w) an N-dimensional vector either in $\mathbb{B}^N$ or $\mathbb{H}^N$ with $z \in \mathbb{C}$ being the first component and $w \in \mathbb{C}^{N-1}$ being the last $N-1$ components.

Forward iteration in the unit ball of $\mathbb{C}^N$ in the hyperbolic case was studied in [5] and [6]. In [6] the Schröder equation (1.2) was solved with ψ being holomorphic map $\psi : \mathbb{B}^N \to \mathbb{H}$ given some additional conditions. In [5], f was conjugated to its linear part, assuming some regularity at the Denjoy-Wolff point. Conjugations for elliptic maps were given in [7]. Linearization results for the large class of hyperbolic and parabolic maps of $\mathbb{B}^2$ were proved in [3].

2.2. **Main results.** My main goal is to study backward iterates in the unit ball $\mathbb{B}^N$. The following results are generalizations of Theorem 1.3 and Theorem 1.4 to higher dimensions.

Theorem 2.4. *Let f be a holomorphic self-map of $\mathbb{B}^N$ of hyperbolic or elliptic type with Denjoy-Wolff point p. Let $\{Z_n\}$ be a backward-iteration sequence for f with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:*

(1) *There exists a point $q \in \partial\mathbb{B}^N$, $q \neq p$, such that $Z_n \to q$ as n tends to infinity,*
(2) *$\{Z_n\}$ stays in a Koranyi region with vertex q,*
(3) *Julia's lemma (1.4) holds for q with a finite multiplier $\alpha \geq \frac{1}{c}$, where $c < 1$ is a constant that depends on f.*

Remark 2.5. In the hyperbolic case, c is the multiplier at p, see (2.2).

Because of the last statement of the Theorem (2.4), the multiplier $\alpha > 1$, and thus we can introduce the following

Definition 2.6. The point $q \in \partial\mathbb{B}^N$ is called a boundary repelling fixed point (BRFP) for f, if (1.4) holds for some $\alpha > 1$.

Remark 2.7. It follows from Julia's lemma (Theorem 2.1) that the above definition of multiplier is equivalent to (2.1).

Definition 2.8. The Koranyi region $K(q, M)$ of vertex $q \in \partial\mathbb{B}^N$ and amplitude $M > 1$ is the set

$$K(q, M) = \left\{ Z \in \mathbb{B}^N : \frac{|1 - (Z, q)|}{1 - \|Z\|} < M \right\}. \tag{2.3}$$

Koranyi regions are natural generalizations of the Stolz regions in $\mathbb{D}$ and can be used to define K-limits:

Definition 2.9. We will say that function f has K-limit λ at $q \in \partial\mathbb{B}^N$ if for any $M > 1$ $f(Z) \to \lambda$ as $Z \to q$ within $K(q, M)$.

In one dimension this is exactly the non-tangential limit, while when $N > 1$ the approach is restricted to be non-tangential only in the the radial dimension, see [1].

Theorem 2.10. *Suppose f is an analytic function of $\mathbb{H}^N$ with $f(\mathbb{H}^N) \subseteq \mathbb{H}^N$ and 0 is a boundary repelling fixed point for f with multiplier $1 < \alpha < \infty$, isolated from other boundary repelling fixed points with multipliers less or equal to α. Consider the automorphism of $\mathbb{H}^N$: $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$. Then there is an analytic map ψ of $\mathbb{H}^N$ with $\psi(\mathbb{H}^N) \subseteq \mathbb{H}^N$ and $\psi(z, w) = \psi(z, 0)$, which has restricted K-limit 0 at 0 (see [1]), such that*

$$\psi \circ \eta(Z) = f \circ \psi(Z), \tag{2.4}$$

for every $Z \in \mathbb{H}^N$.

It follows from the proof of Theorem 2.10, that every isolated boundary repelling fixed point is a limit of some backward-iteration sequence with bounded hyperbolic step. Thus in the hyperbolic and elliptic cases we have the following characterization of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if a BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

The intertwining map ψ in Theorem 2.10 satisfies $\psi(z, w) = \psi(z, 0)$ and essentially is a map from one dimensional subspace of $\mathbb{H}^N$ to $\mathbb{H}^N$, therefore that conjugation does not provide information about behavior of f outside of one dimensional image of ψ. It then is natural to identify situations in which we can find a conjugation such that the image of the intertwining map ψ has larger dimension.

Theorem 2.11. *Let f be expandable at 0 (see [5]) and 0 be a boundary repelling fixed point with multiplier $1 < \alpha < \infty$. Assume further that the matrix A in the definition of f is diagonal, and without loss of generality let its eigenvalues be $a_{j,j} = \sqrt{\alpha} e^{i\theta_j}$ for $j = 1 \dots L$ (L is an integer, $0 \le L \le N-1$) and $|a_{j,j}|^2 < \alpha$ for $j = L+1 \dots N-1$. Define Ω as a diagonal matrix with $\Omega_{j,j} = e^{i\theta_j}$ for $j = 1 \dots L$ and $\Omega_{j,j} = 1$ for $j = L+1 \dots N-1$. Then the conjugation (2.4) holds for $\eta(z, w) = \big(\alpha z, \Omega \alpha^{1/2} w\big)$ and intertwining map ψ such that $\psi(z, w) = \psi(p_L(z, w))$, where p_L is a projection on the first $L+1$ dimensions.*

2.3. **Examples.** It follows from a theorem of Cowen-Pommerenke [8], that the unit disk $\mathbb{D}$ contains at most finitely many BRFP's with multiplier less or equal to a fixed $\alpha < \infty$. It is no longer true in higher dimension. Below I will provide some new examples, in particular, functions in the two-dimensional Siegel domain that have non-isolated BRFPs.

Example 2.12 (Example of a quadratic function with non-isolated BRFP)**.** Consider the function $f(z,w) := (2z + w^2, w)$. Then $f^{\circ n}(z,w) = (2^n z + (2^n - 1)w^2, w)$, the Denjoy-Wolff point is infinity and this is the hyperbolic case. The curve $\{(r^2, ir) | r \in \mathbb{R}\}$ is clearly the set of fixed points on the boundary, all of them having the same multiplier $\alpha = 2$, and neither of them is isolated.

Remark 2.13. Though $(0,0)$ is non-isolated BRFP for $f(z,w) := (2z + w^2, w)$, $Z_n = (\frac{1}{2^n}, 0)$ is clearly an example of backward-iteration sequence with step $d = \frac{1}{3}$, and it is still possible to construct a conjugation as in Theorem 2.10.

Example 2.14. Let $\phi : \mathbb{H} \to \mathbb{H}$ be a holomorphic function of right-hand side half-plane, of hyperbolic or parabolic type, with the Denjoy-Wolff point infinity. Define a function f on $\mathbb{H}^2$ as $f(z,w) := (\phi(z - w^2) + w^2, w)$. This function maps $\mathbb{H}^2$ into itself, infinity is the Denjoy-Wolff point for f and f has the same type and same multiplier at infinity as ϕ. Moreover, if ϕ has a BRFP $y_0 i \in \partial\mathbb{H}$ then f has a 1-dimensional real submanifold $\{(y_0 i + t^2, t) | t \in \mathbb{R}\}$ of BRFPs.

3. Open questions and future goals

3.1. **The dimension of the stable set.** The stable set $\mathcal{S}$ at the BRFP q is defined as the union of all backward-iteration sequences with bounded pseudo-hyperbolic step that tend to q. In one dimension, $\mathcal{S} = \psi(\mathbb{H})$. It is important to understand the properties of the stable set in N dimensions, because it may help to find the "best possible" intertwining map, i.e. the intertwining map whose image has the largest dimension.

3.2. **Non-isolated fixed points and necessary conditions for conjugation at BRFP.** As we can see from Remark 2.13, the condition on the BRFP to be isolated is sufficient, but not necessary. It is still not known if there are any BRFP for which the conjugation construction does not work. One needs to prove a result, similar to Theorem 2.10 for non-isolated BRFP or to find necessary conditions on BRFP so that the conjugation construction will work.

3.3. **Uniqueness of the intertwining map.** In the forward-iteration case the intertwining map in (1.2) is unique. It will be interesting to understand uniqueness better for the backward iterates even in one dimension, since in [13] the is uniqueness stated only for maps which are semi-conformal at BRFP.

3.4. **Parabolic case.** The parabolic case is the least understood case even in one dimension. While forward iterates of a parabolic non-zero-step function converge to the Denjoy-Wolff point tangentially (Remark 1, [15]), in the parabolic zero-step case, forward iterates may converge radially, but a complete classification of their behavior has still not been achieved. The other interesting question is whether zero-step and non-zero-step cases are well-defined in several variables.

3.5. **Conjugations in several variables.** While some linearization results for hyperbolic and parabolic maps were proved ([6], [9] and [3]), they require some additional conditions and thus valid for specific classes of maps. It might be possible to generalize classical conjugation results for different class of maps or even in full generality.

3.6. **Composition operators.** In one dimension the conjugation of f is connected to spectral properties of the composition operator $C_f(g) = g \circ f$, see [14]. Similar theory may be developed in the ball.

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