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RESEARCH STATEMENT

I study properties of backward iterates of analytic self-maps of the unit ball $\mathbb{B}^N$ in $\mathbb{C}^N$. Many facts were established about such maps in the 1-dimensional case (i.e. for self-maps of the unit disk), and I focus on generalizing some of them in higher dimension.

1. Boundary repelling fixed points

Let f be analytic self-map of the unit disk $\mathbb{D}$. By Schwarz's lemma, the map f may have at most one fixed point in $\mathbb{D}$. By Denjoy-Wolff theorem, the sequence of forward iterates of f must converge uniformly on compact subsets of $\mathbb{D}$ to the point $p \in \overline{\mathbb{D}}$. The point p is called the Denjoy-Wolff point of f. When $p \in \mathbb{D}$, f is called elliptic, when $p \in \partial\mathbb{D}$ and $f'(p) < 1$, hyperbolic, and when $p \in \partial\mathbb{D}$ and $f'(p) = 1$, parabolic.

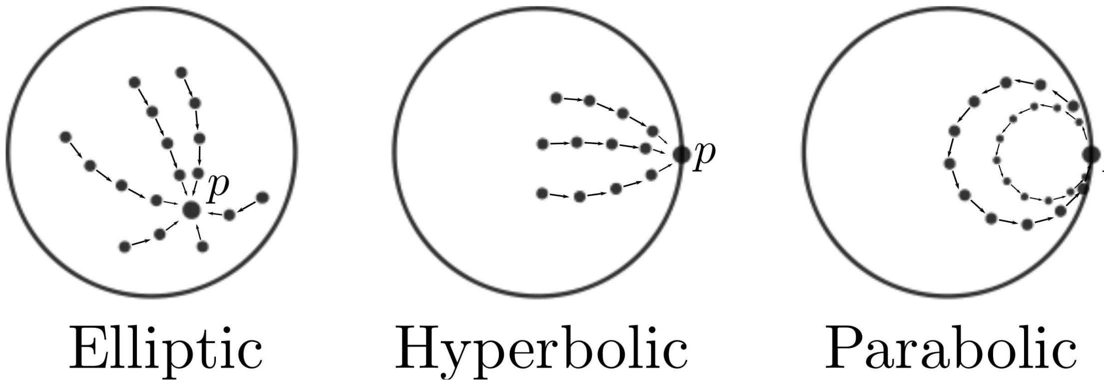

Figure 1. Denjoy-Wolff point p and typical orbits in elliptic, hyperbolic and parabolic cases.

It follows that every other fixed point q of f must lie on $\partial\mathbb{D}$, and its multiplier $f'(q) > 1$. Such points are called boundary repelling fixed points (BRFP) and they have several nice properties, studied in [4] and [3]; in particular, they are limits of backward-iteration sequences for f (i.e. sequences $\{w_n\}_{n=0}^{\infty}$ such that $f(w_{n+1}) = w_n$) with bounded hyperbolic step.

Theorem 1.1 (Poggi-Corradini, [4]). *Let $\{w_n\}_{n=0}^{\infty}$ be a backward-iteration sequence for analytic self-map of the disk f with bounded pseudo-hyperbolic step $d(w_n, w_{n+1}) \leq a < 1$. Then:*

1. There is a point $q \in \partial\mathbb{D}$ such that $w_n \to q$ as n tends to infinity, and q is a fixed point for f with a well-defined multiplier $f'(q) < \infty$

2. If $q \neq p$, then q is a BRFP. If $q = p$, f is necessarily of parabolic type.

3. When q is a BRFP, w_n tends to q along a non-tangential direction.

4. If $q = p$ (in the parabolic case), then w_n tend to q tangentially.

I have generalized theorem 1.1 for higher dimension when f is elliptic or hyperbolic.

Theorem 1.2. *Let f be a holomorphic self-map of $\mathbb{B}^N$ of hyperbolic or elliptic type, $\{Z_n\}$ be a backward-iteration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:*

1. There exists a point $\partial\mathbb{B}^N \ni \tau \neq p$ such that $Z_n \xrightarrow[n\to\infty]{} \tau$.

2. $\{Z_n\}$ stays in a Koranyi region at τ.

3. Julia's lemma holds for τ with multiplier $A \geq \frac{1}{c}$, where $c < 1$ is a constant that depends on f.

2. Conjugations at BRFP

My next goal was to find a conjugation for f near τ and study its properties. Conjugations are a powerful tool to describe the dynamics of f near a fixed point. In [7] and [6], the existence of analytic conjugation maps was proved for forward iteration in $\mathbb{C}$. The multi-dimensional case of forward iterations was studied in [2]; and a partial (one-dimensional) conjugation was found. In [4], a conjugation was constructed for boundary repelling fixed points in the unit disk. My goal was to find an automorphism η of the unit ball and an analytic map ψ with $\psi(\mathbb{B}^N) \subseteq \mathbb{B}^N$, with some regularity at the BRFP τ, such that

$$\psi \circ \eta = f \circ \eta \tag{2.1}$$

everywhere in $\mathbb{B}^N$. I follow the approach of [4] to produce a conjugation and intertwining map, using the sequence of iterates of f together with some renormalization. The partial limit of such iterations is a candidate for the intertwining map ψ. I formulate the result in Siegel domain $\mathbb{H}^N$, which is biholomorphically equivalent to the unit ball $\mathbb{B}^N$.

Theorem 2.1. *Suppose f is an analytic self-map of $\mathbb{H}^N$ and 0 is a BRFP for f with multiplier $1 < \alpha < \infty$, isolated from the other BRFP's with multipliers less or equal to α. Let $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$ be an automorphism of $\mathbb{H}^N$. Then there is an analytic map ψ of $\mathbb{H}^N$ with $\psi(\mathbb{H}^N) \subseteq \mathbb{H}^N$ and $\psi(z, w) = \psi(z, 0)$, which has restricted K-limit 0 at 0, such that (2.1) holds.*

Remark 2.2. It follows from the proof of Theorem 2.1, that every isolated BRFP is a limit of some backward-iteration sequence with bounded hyperbolic step. Thus in the hyperbolic and elliptic cases we have the following characterization of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP (possibly not isolated); and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.

The intertwining map ψ from the statement of Theorem 2.1 satisfies $\psi(z,w)=\psi(z,0)$ and essentially is a map from one dimensional subspace of $\mathbb{H}^N$ to $\mathbb{H}^N$, therefore that conjugation does not provide information about behavior of f outside of one dimensional image of ψ. It is possible to find better conjugation such that the image of the intertwining is a manifold in $\mathbb{H}^N$ of higher dimension, if the function f satisfies some regularity condition at BRFP. Such functions (we will call them expandable) were defined in [1].

Theorem 2.3. *Let f be expandable at 0, and 0 be a boundary repelling fixed point with multiplier $1<\alpha<\infty$. Assume further that the matrix A in the expansion of f is diagonal, and without loss of generality let its eigenvalues be $a_{j,j}=\sqrt{\alpha}e^{i\theta_j}$ for $j=1\dots K$ and $|a_{j,j}|^2<\alpha$ for $j=K+1\dots N-1$. Define Ω as a diagonal matrix with $\Omega_{j,j}=e^{i\theta_j}$ for $j=1\dots K$ and $\Omega_{j,j}=1$ for $j=K+1\dots N-1$. Then the conjugation (2.1) holds for $\eta(z,w)=\left(\alpha z,\Omega\alpha^{1/2}w\right)$ and intertwining map ψ such that $\psi(z,w)=\psi(p_K(z,w))$, where p_K is a projection on the first $K+1$ dimensions.*

3. Examples

Here I will provide some new examples, in particular, functions in the two-dimensional Siegel domain that have non-isolated BRFPs, a phenomenon that never occurs in one dimension.

Example 3.1 (Example of a quadratic function with non-isolated BRFP)**.** Consider the function $f(z,w):=(2z+w^2,w)$. Then $f^{\circ n}(z,w)=(2^nz+(2^n-1)w^2,w)$, the Denjoy-Wolff point is infinity and this is the hyperbolic case. The curve $\{(r^2,ir)|r\in\mathbb{R}\}$ is clearly the set of fixed points on the boundary, all of them having the same multiplier $\alpha=2$, and neither of them is isolated.

Example 3.2. Let $\phi:\mathbb{H}\to\mathbb{H}$ be a holomorphic function of right-hand side half-plane, of hyperbolic or parabolic type, with the Denjoy-Wolff point infinity. Define a function f on $\mathbb{H}^2$ as $f(z,w):=(\phi(z-w^2)+w^2,w)$. This function maps $\mathbb{H}^2$ into itself, infinity is the Denjoy-Wolff point for f and f has the same type and same multiplier at infinity as ϕ. Moreover, if ϕ has a BRFP $y_0i\in\partial\mathbb{H}$ then f has a 1-dimensional real submanifold $\{(y_0i+t^2,t)|t\in\mathbb{R}\}$ of BRFPs.

4. Open questions and future goals

4.1. **The dimension of the stable set.** The stable set $\mathcal{S}$ at the BRFP q is defined as the union of all backward-iteration sequences with bounded pseudo-hyperbolic step that tend to q. In one dimension, $\mathcal{S}=\psi(\mathbb{H})$. It is important to understand the properties of the stable set in N dimensions, because it may help to find the "best possible" intertwining map, i.e. the intertwining map whose image has the largest dimension.

4.2. **Non-isolated fixed points and necessary conditions for conjugation at BRFP.** It is still possible to construct a backward iteration sequence with bounded step and, consequently, conjugations for BRFPs in Examples 3.1 and 3.2, thus the condition on the BRFP to be isolated is sufficient, but not necessary. It is still not known if there are any BRFP for which the conjugation construction does not work. One needs to prove a result, similar to Theorem 2.1 for non-isolated BRFP or to find necessary conditions on BRFP so that the conjugation construction will work.

4.3. **Uniqueness of the intertwining map.** In the forward-iteration case the intertwining map is unique. It will be interesting to understand uniqueness better for the backward iterates even in one dimension, since in [4] the is uniqueness stated only for maps which are semi-conformal at BRFP.

4.4. **Parabolic case.** Theorem 1.2 generalizes the one-dimensional Theorem 1.1 only in hyperbolic and elliptic cases. It is still not known whether backward-iteration sequences with bounded step always converge for parabolic maps in higher dimensions.

4.5. **Composition operators.** In one dimension the conjugation of f is connected to spectral properties of the composition operator $C_f(g) = g \circ f$, see [5]. Similar theory may be developed in the ball.

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