Backward iteration in the unit ball

Olena Ostapyuk
Kansas State University

http://arxiv.org/abs/0910.5451
One-dimensional case

Forward iteration

Let $f$ be analytic self-map of $\mathbb{D} = \{z : |z| < 1\}$

$n$-th iterate of $f$ $f_n = f \circ \ldots \circ f$ \(n \text{ times}\)

By **Schwarz's lemma**, $f$ is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \bar{w}z} \right|$$

**Theorem (Denjoy-Wolff)**

If a self-map of the disk $f$ is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to $p$.

if $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$

if $p \in \partial \mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits

The point $p$ is called the **Denjoy-Wolff point** of $f$. 
Cases:

1. $p \in \mathbb{D}$, $f$ is called elliptic

2. $p \in \partial \mathbb{D}$, $f'(p) < 1$ hyperbolic

3. $p \in \partial \mathbb{D}$, $f'(p) = 1$ parabolic

If $p \in \partial \mathbb{D}$, Julia’s lemma holds for the point $p$, and multiplier $c = f'(p) \leq 1$:

$$\forall R > 0 \quad f(H(p, R)) \subseteq H(p, cR),$$

where $H(p, R)$ is a horocycle at $p \in \partial \mathbb{D}$ of radius $R$:

$$H(p, R) := \left\{ z \in \mathbb{D} : \frac{|p - z|^2}{1 - |z|^2} < R \right\}$$
Backward iteration

Backward-iteration sequence:
\[ \{z_n\}_{n=0}^{\infty}, \quad f(z_{n+1}) = z_n \quad \text{for} \quad n = 0, 1, 2 \ldots \]

The sequence \( d(z_n, z_{n+1}) \) is increasing, so we need a bound on the pseudo-hyperbolic step:
\[ d(z_n, z_{n+1}) \leq a < 1 \]

**Theorem (Poggi-Corradini, 2003)**

Let \( \{z_n\}_{n=0}^{\infty} \) be a backward-iteration sequence for analytic self-map of the disk \( f \) with bounded pseudo-hyperbolic step \( d(z_n, z_{n+1}) \leq a < 1 \). Then:

1. \( z_n \rightarrow q \in \partial \mathbb{D} \), and \( q \) is a fixed point with a well-defined multiplier \( f'(q) < \infty \)

2. If \( q \neq p \), then \( q \) is a **boundary repelling fixed point** (BRFP) (i.e. \( f'(q) > 1 \)). If \( q = p \), \( f \) is of parabolic type.

3. When \( q \) is BRFP, the convergence \( z_n \rightarrow q \) is non-tangential.

4. If \( q = p \), then \( w_n \rightarrow q \) tangentially.
Multi-dimensional case

\( \mathbb{C}^N \), inner product \((Z,W) = \sum_{j=1}^{N} Z_j \overline{W_j}\)

\( ||Z||^2 = (Z,Z) \)

Unit ball \( \mathbb{B}^N = \{Z \in \mathbb{C}^N : ||Z|| < 1\} \)

**Julia’s lemma** in \( \mathbb{B}^N \): Let \( f \) be a holomorphic self-map of \( \mathbb{B}^N \) and \( X \in \partial \mathbb{B}^N \) such that

\[
\liminf_{Z \to X} \frac{1 - ||f(Z)||}{1 - ||Z||} = \alpha < \infty
\]

Then there exists a unique \( Y \in \partial \mathbb{B}^N \) such that \( \forall R > 0 \ f(H(X,R)) \subset H(Y,\alpha R). \)

**Horosphere** of center \( X \in \partial \mathbb{B}^N \) and radius \( R > 0 \):

\[
H(X,R) = \left\{ Z \in \mathbb{B}^N : \frac{|1 - (Z,X)|^2}{1 - ||Z||^2} < R \right\}
\]
Multi-dimensional version of Denjoy-Wolff theorem holds:

**Theorem (MacCluer, 1983)**

If $f$ has no fixed points in $\mathbb{B}^N$, then $f_n$ converges uniformly on compacta to $p \in \partial \mathbb{B}^N$, the number $c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1]$ is a multiplier of $f$ at $p$.

$f$ is called hyperbolic if $c < 1$ and parabolic if $c = 1$.

**Siegel domain:**

$\mathbb{H}^N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \text{Re}z > \|w\|^2\}$

Cayley transform: $\mathcal{C} : \mathbb{B}^N \to \mathbb{H}^N$

$\mathcal{C}((z, w)) = \left( \frac{1 + z}{1 - z}, \frac{w}{1 - z} \right)$

$\mathcal{C}^{-1}((z, w)) = \left( \frac{z - 1}{z + 1}, \frac{2w}{z + 1} \right)$
**Theorem 1.** Let $f$ be a analytic self-map of $\mathbb{B}^N$ of hyperbolic type (with Denjoy-Wolff point $p \in \partial \mathbb{B}^N$), $\{Z_n\}$ be a backward-iteration sequence with bounded pseudo-hyperbolic step $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \leq a < 1$. Then:

1. There exists a point $\partial \mathbb{B}^N \ni \tau \neq p$ such that $Z_n \xrightarrow{n \to \infty} \tau$

2. $\{Z_n\}$ stays in a Koranyi region

3. Julia's lemma holds for $\tau$ with multiplier $\alpha \geq \frac{1}{c}$, where $c$ is the multiplier at $p$.

Since $\alpha \geq \frac{1}{c} > 1$, the point $q \in \partial \mathbb{B}^N$ is called the **boundary repelling fixed point** (BRFP) for $f$.

**Characterization** of BRFP in terms of backward-iteration sequences: Every backward-iteration sequence with bounded hyperbolic step converges to a BRFP; and if BRFP is isolated, then we can construct a backward-iteration sequence with bounded hyperbolic step that converges to it.
Conjugations

1-dimensional case, forward iteration (Valiron, 1931):  
\[ \psi \circ f = \frac{1}{c} \psi, \]
where \( \psi : \mathbb{D} \rightarrow \mathbb{H} \) is an analytic map to a half-plane.

1-dimensional case, backward iteration (Poggi-Corradini, 2000): an analytic self-map of the unit disc \( f \) with BRFP \( 1 \in \partial \mathbb{D} \) and multiplier \( \alpha \) at 1 can be conjugated to the automorphism \( \eta(z) = (z - a)/(1 - az) \), where \( a = (\alpha - 1)/(\alpha + 1) \):

\[ \psi \circ \eta(z) = f \circ \psi(z), \]
via an analytic map \( \psi \) of \( \mathbb{D} \) with \( \psi(\mathbb{D}) \subseteq \mathbb{D} \), which has non-tangential limit 1 at 1.

N-dimensional case, forward iteration (Bracci, Gentili, Poggi-Corradini): conjugation to a multiplication via \( \psi : \mathbb{B}^N \rightarrow \mathbb{H} \).

(Bracci, Gentili, 2005): \( f \) is conjugated to its linear part, assuming some regularity at the Denjoy-Wolff point.
Theorem 2. Suppose $f : \mathbb{H}^N \rightarrow \mathbb{H}^N$ is an analytic function of hyperbolic type and 0 is an isolated boundary repelling fixed point for $f$ with multiplier $1 < \alpha < \infty$. Then $f$ is conjugated to the automorphism $\eta(z, w) = (\alpha z, \sqrt{\alpha} w)$

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map $\psi$.

Construction of $\psi$:

$$\psi = \lim_{n \to \infty} \{f_n \circ \tau_n \circ p_1\}$$

where $p_1(z, w) := (z, 0)$ is the projection on the first (radial) dimension, so

$$\psi(z, w) = \psi(z, 0)$$

and is essentially one-dimensional map.
Conjugation for expandable maps

Definition: An analytic map $f : \mathbb{H}^N \to \mathbb{H}^N$ is called \textbf{expandable} at 0 if

$$f(z, w) = (\alpha z + o(|z|), Aw + o(|z|^{1/2}))$$

In particular, 0 is a fixed point of $f$ and $\alpha$ is the multiplier of $f$ at 0.

\textbf{Theorem 3.} Let $f$ be expandable at 0, of hyperbolic type, and let the matrix $A$ be diagonal, and WLOG

$$|a_{j,j}| = \sqrt{\alpha} \text{ for } j = 1 \ldots L$$

$$|a_{j,j}| < \sqrt{\alpha} \text{ for } j = L + 1 \ldots N - 1.$$ 

Then $f$ is conjugated to the automorphism $\eta(z, w) = (\alpha z, \Omega \sqrt{\alpha} w)$ ($\Omega$ is a rotation):

$$\psi \circ \eta(Z) = f \circ \psi(Z),$$

via an analytic intertwining map $\psi(z, w) = \psi(p_L(z, w))$, where $p_L$ is a projection on the first $L + 1$ dimensions.
Open questions

1. "Best possible" intertwining map

2. Number of BRFP

3. Uniqueness of the intertwining map

4. Parabolic and "elliptic" cases