Parabolic dynamics in the disk and in the ball

Olena Ostapyuk

Department of Mathematics
University of Northern Iowa

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Outline of my Talk

1. Introduction
2. Parabolic case in the disk
3. Parabolic maps of the ball
4. Examples and Special Cases
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4. Examples and Special Cases
Let $f$ be analytic self-map of $\mathbb{D} = \{ z : |z| < 1 \}$

n-th iterate of $f$ $f_n = f \circ \ldots \circ f$

By Schwarz’s lemma, $f$ is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \frac{|z - w|}{1 - \overline{w}z}$$

**Theorem (Denjoy-Wolff)**

If a self-map of the disk $f$ is not an elliptic automorphism, then there exist a unique point $p \in \mathbb{D}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to $p$.

If $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$

If $p \in \partial \mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits
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Let $f$ be analytic self-map of $\mathbb{D} = \{ z : |z| < 1 \}$

n-th iterate of $f$ $f_n = f \circ \ldots \circ f$ \(n \) times

By **Schwarz’s lemma**, $f$ is a contraction in the pseudo-hyperbolic metric

$$d(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right|$$

**Theorem (Denjoy-Wolff)**

If a self-map of the disk $f$ is not an elliptic automorphism, then there exist a unique point $p \in \overline{\mathbb{D}}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to $p$.

If $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$

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*If a self-map of the disk $f$ is not an elliptic automorphism, then there exist a unique point $p \in \mathbb{D}$ such that the sequence $f_n(z)$ converges uniformly on compact subsets to $p$. If $p \in \mathbb{D}$, then $f(p) = p$ and $|f'(p)| < 1$ if $p \in \partial \mathbb{D}$, then $f(p) = p$ and $0 < f'(p) \leq 1$ in the sense of non-tangential limits*
The point $p$ is called the **Denjoy-Wolff point** of $f$.

Cases:
1. $p \in \mathbb{D}$ $f$ is called elliptic
2. $p \in \partial \mathbb{D}$, $f'(p) < 1$ hyperbolic
3. $p \in \partial \mathbb{D}$, $f'(p) = 1$ parabolic
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![Elliptic, Hyperbolic, Parabolic diagrams](image.png)
Parabolic case in the disk (or half-plane)

Consider a forward orbit

\[ z_n = f_n(z_0) := f \circ \ldots \circ f(z_0) \]

By Schwarz’s lemma \( d(z_n, z_{n+1}) \leq d(z_{n-1}, z_n) \), and the pseudo-hyperbolic step \( d_n := d(z_n, z_{n+1}) \) must have a limit:

\[ d_n \xrightarrow{n \to \infty} b \]

**Definition**

We will call a sequence \( \{z_n\} \) a zero step (resp. non-zero step) sequence if \( b = 0 \) (resp. \( b > 0 \)).

Another model: right-half plane \( \mathbb{H} := \{z \mid \text{Re } z > 0\} \), biholomorphically equivalent to the unit disk \( \mathbb{D} \).
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**Theorem (Pommerenke)**

Consider $f$ a parabolic self-map of $\mathbb{H}$ with Denjoy-Wolff point $\infty$, and define $z_n = x_n + iy_n := f_n(1)$,

$$g_n(z) := \frac{f_n(z) - iy_n}{x_n}.$$

Then the limit $g(z) = \lim_{n \to \infty} g_n(z)$ exists locally uniformly and

$$g(f(z)) = \phi(g(z)) \quad \forall z \in \mathbb{H},$$

and

- $\phi(z) = z + ib$ (vertical translation) if $\{z_n\}$ has non-zero step;
- $\phi(z) = z$ and $g(z) \equiv 1$ (trivial conjugation) if $\{z_n\}$ has zero step.
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Corollary 1.

The step does not depend on the choice of the sequence and depends on map only; i.e. for a given parabolic map either all orbits have zero step or all orbits have non-zero step.

Thus we can classify parabolic maps of the disk (or a half-plane) as parabolic zero-step and parabolic non-zero step maps.

Corollary 2.

In parabolic non-zero-step case in \( \mathbb{H} \),

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\arg z_n \xrightarrow{n \to \infty} \pm \frac{\pi}{2}
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i.e. orbits converge to the Denjoy-Wolff point tangentially to the boundary.
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Theorem (Baker, Pommerenke)

Let $f$ be parabolic zero-step map of $\mathbb{H}$ with Denjoy-Wolff point infinity, then there exists $h : \mathbb{H} \to \mathbb{C}$ such that

$$h(f(z)) = h(z) + 1 \quad \forall z \in \mathbb{H},$$

i.e. $f$ is conjugated to a horizontal shift in the plane.

Orbits in parabolic zero-step case may converge tangentially as well as non-tangentially.

Conjecture 1.

Let $f$ be a parabolic zero-step map of $\mathbb{H}$ with Denjoy-Wolff point infinity, then there exists direction $\theta \in [-\pi/2, \pi/2]$ such that for any orbit $\{z_n\}$

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Multi-dimensional case

$f$ is self-map of $N$-dimensional unit ball $\mathbb{B}^N = \{Z \in \mathbb{C}^N : \|Z\| < 1\}$.

Schwarz’s lemma still holds in $\mathbb{B}^N$, with pseudo-hyperbolic distance defined as

$$d_{\mathbb{B}^N}(Z, W) := \left( \frac{|1 - \langle Z, W \rangle|^2}{(1 - \|Z\|^2)(1 - \|W\|^2)} \right)^{1/2}.$$

Multi-dimensional version of Denjoy-Wolff theorem holds:

**Theorem (Hervé, MacCluer, 1983)**

If $f$ has no fixed points in $\mathbb{B}^N$, then $f_n$ converges uniformly on compacta to $p \in \partial \mathbb{B}^N$, the number $c := \liminf_{Z \to p} \frac{1 - \|f(Z)\|}{1 - \|Z\|} \in (0, 1]$ is a multiplier of $f$ at $p$.

$f$ is called **hyperbolic** if $c < 1$ and **parabolic** if $c = 1$. 
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\( f \) is self-map of \( N \)-dimensional unit ball \( \mathbb{B}^N = \{ Z \in \mathbb{C}^N : \|Z\| < 1 \} \).

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An analog of the half-plane $\mathbb{H}$ in several dimensions is

**Siegel domain** (or Siegel half-space)

$$\mathbb{H}^N = \{(z, w) \in \mathbb{C} \times \mathbb{C}^{N-1} : \Re z > \|w\|^2\},$$

which is biholomorphically equivalent to $\mathbb{B}^N$ via

**Cayley transform:**

$$C : \mathbb{B}^N \rightarrow \mathbb{H}^N$$

$$C((z, w)) = \left(\frac{1+z}{1-z}, \frac{w}{1-z}\right) \quad C^{-1}((z, w)) = \left(\frac{z-1}{z+1}, \frac{2w}{z+1}\right).$$
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For parabolic maps of the ball, zero and non-zero step cases are well-defined only for sequences.

The question whether the same map can have sequences of both types is still open.

**Conjecture 2.**

Let $f$ a self map of $\mathbb{B}^N$ of parabolic type. If the step $d_{\mathbb{B}^N}(f_n(Z_0), f_{n+1}(Z_0)) \to 0$ for some $Z_0 \in \mathbb{B}^N$, then $d_{\mathbb{B}^N}(f_n(Z), f_{n+1}(Z)) \to 0$ for all $Z \in \mathbb{B}^N$. 
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Definition

The Koranyi region $K(X, M)$ of vertex $X \in \partial \mathbb{B}^N$ and amplitude $M > 1$ is the set

$$K(X, M) = \left\{ Z \in \mathbb{B}^N \left| \frac{1 - \langle Z, X \rangle}{1 - \|Z\|} < M \right. \right\}.$$

When $N = 1$, it is the usual Stolz angle in the disk; but for $N > 1$ the region is tangent to the boundary of the ball along some directions.

Definition

For $X \in \partial \mathbb{B}^N$, a sequence $Z_n \rightarrow X$ is called special if

$$\lim_{n \rightarrow \infty} \frac{\|Z_n - \langle Z_n, X \rangle X\|^2}{1 - \|\langle Z_n, X \rangle X\|^2} = 0,$$

and restricted if it is special and its orthogonal projection $\langle Z_n, X \rangle X$ is non-tangential.
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non-tangential $\Rightarrow$ restricted $\Rightarrow$ lies in a Koranyi region

**Theorem (O.O.)**

*If the sequence of forward iterates $\{Z_n\}_{n=1}^{\infty}$ for parabolic self-map of the unit ball is restricted, then it must have zero step, i.e. $d_{\mathbb{B}^N}(Z_n, Z_{n+1}) \to 0$.***

In particular, every non-zero-step sequence must converge tangentially.
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Known examples

of parabolic maps in $\mathbb{H}^N$ are:

Example 1: **Heisenberg translations**

$$(z, w) \mapsto (z + z_0 + 2\langle w, w_0 \rangle, w + w_0) \text{ for some } (z_0, w_0) \in \partial \mathbb{H}^N, \text{ i.e.} \quad \text{Re } z_0 = \|w_0\|^2.$$ 

They are parabolic automorphisms of $\mathbb{H}^N$ and thus have non-zero step.

Example 2: **Generalized Heisenberg translations**

$$(z, w) \mapsto (z + z_0 + 2\langle w, w_0 \rangle, w + w_0) \text{ with } \text{Re } z_0 \geq \|w_0\|^2.$$ 

They have zero step unless $\text{Re } z_0 = \|w_0\|^2.$
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**Example 2: Generalized Heisenberg translations**

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They have zero step unless $\Re z_0 = \| w_0 \|^2.$
Example 3: **Parabolic linear-fractional maps of** $\mathbb{H}^N$

*(linear-fractional self-maps of the ball, transferred to $\mathbb{H}^N$).

$$f(Z) := \frac{AZ + B}{\langle Z, \overline{C} \rangle + d}$$

with $f(\mathbb{B}^N) \subseteq \mathbb{B}^N$, where $A$ is $N \times N$-matrix, $B, C \in \mathbb{C}^N$ and $d \in \mathbb{C}$.

**Theorem (Bayart)**

Parabolic linear-fractional maps that do not fix any non-trivial affine subset of $\mathbb{B}^N$ are conjugated to generalized Heisenberg translations.
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Example 4. (O.O.):

Given one-dimensional $\phi : \mathbb{H} \to \mathbb{H}$ of hyperbolic or parabolic type, with the Denjoy-Wolff point $\infty$,

construct $f(z, w) := (\phi(z - w^2) + w^2, w)$. Then:

$f$ is the self-map of $\mathbb{H}^2$ with the Denjoy-Wolff point $\infty$ and has the same type and same multiplier at $\infty$ as $\phi$.

Moreover, all forward orbits have zero (resp. non-zero) step, if $\phi$ is parabolic zero (resp. non-zero) step map.
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Given one-dimensional \( \phi : \mathbb{H} \to \mathbb{H} \) of hyperbolic or parabolic type, with the Denjoy-Wolff point \( \infty \), construct \( f(z, w) := (\phi(z - w^2) + w^2, w) \). Then:

\( f \) is the self-map of \( \mathbb{H}^2 \) with the Denjoy-Wolff point \( \infty \) and has the same type and same multiplier at \( \infty \) as \( \phi \).

Moreover, all forward orbits have zero (resp. non-zero) step, if \( \phi \) is parabolic zero (resp. non-zero) step map.
Given one-dimensional $\phi : \mathbb{H} \to \mathbb{H}$ of hyperbolic or parabolic type, with the Denjoy-Wolff point $\infty$, construct $f(z, w) := (\phi(z - w^2) + w^2, w)$. Then: $f$ is the self-map of $\mathbb{H}^2$ with the Denjoy-Wolff point $\infty$ and has the same type and same multiplier at $\infty$ as $\phi$.
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Given one-dimensional $\phi : \mathbb{H} \rightarrow \mathbb{H}$ of hyperbolic or parabolic type, with the Denjoy-Wolff point $\infty$,

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Example 5. (O.O.):

Given one-dimensional \( \phi : \mathbb{H} \to \mathbb{H} \) of parabolic type, with the Denjoy-Wolff point \( \infty \), construct

\[
f(z, w) := (\phi(z) + z_0 + 2 \langle w, w_0 \rangle, w + w_0)
\]

for some \( (z_0, w_0) \in \partial \mathbb{H}^N \).

Then \( f \) is the self-map of \( \mathbb{H}^N \) with the Denjoy-Wolff point \( \infty \) of parabolic type.
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Example 6. (Bayart) **Maps with some regularity at the Denjoy-Wolff point**

\[ D^{n+\varepsilon} : \text{parabolic maps of } \mathbb{B}^2 \text{ that can be expanded near the Denjoy-Wolff point up to a certain order.} \]

Depending on \( n \) and the first derivative matrix, they can be conjugated to various generalized Heisenberg translations, in particular:

- If \( n = 5 \) and the matrix is non-diagonalizable, model map is
  \[(z, w) \mapsto (z + z_0 + 2 \langle w, w_0 \rangle, w + w_0)\]
- If \( n = 6 \) and the matrix is diagonalizable, model map is \( z \mapsto z + b \)
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Thank you!