THE LOEWNER PROPERTY


Notations

- $\sigma(x,y)$ is the chordal distance on $\mathbb{C}_\infty$.
- $\Sigma$ is the spherical measure on $\mathbb{C}_\infty$.
- $\Delta(E,F) := \frac{\text{dist}(E,F)}{\min(\text{diam}(E), \text{diam}(F))}$ is the relative distance between sets $E$ and $F$.
- $B(x,r) := \{y : \sigma(y,x) < r\}$ is the open ball of radius $r$ centered at $x$.
- $N_\delta(E) := \{x : \text{dist}(x,E) < \delta\}$ is the $\delta$-neighborhood of $E$.
- $\Gamma(E,F;\Omega)$ is the family of all closed paths in $\Omega$ that connect $E$ and $F$.
- $\text{mod}(\Gamma)$ is the modulus of the path family $\Gamma$.

Definition. A region $\Omega \subseteq \mathbb{C}_\infty$ is called $\phi$-Loewner if there exists a non-increasing $\phi : (0,\infty) \to (0,\infty)$ such that

$$\text{Mod}(\Gamma(E,F;\Omega)) \geq \phi(\Delta(E,F)),$$

for any two disjoint continua $E$ and $F$ in $\overline{\Omega}$.

Equivalent to: Exists $m = m(t) > 0$ such that if $\Delta(E,F) \leq t$ and $\int \rho d\Sigma < m$ then there exists $\gamma \in \Gamma(E,F;\Omega)$ such that $\int_\gamma \rho ds < 1$. (Can take $m(t) := \phi(t)$ for $\phi$-Loewner region and $\phi(s) := \sup\{m(t) : t \geq s\}$.)

1. The unit disk $\mathbb{D}$ is Loewner. An annulus $N_\delta(\mathbb{D}) \setminus \overline{\mathbb{D}}$ with $\delta \in (0,\sqrt{2})$ is $\phi$-Loewner with $\phi = \phi(\delta)$.

2. The image of a Loewner region under quasi-Möbius map is Loewner.

3. A Jordan region bounded by a quasicircle is Loewner.

4. Collar lemma: For a region $\Omega = \mathbb{C}_\infty \setminus \bigcup_{i=1}^n D_i$ with complementary components $D_i$ being $s$-relatively separated closed Jordan regions with boundaries $\partial D_i$ being $k$-quasicircles, it is possible to put a “Loewner collar” $U$ around the smallest complementary component $D_n$ with
thickness proportional to the diameter of $D_n$ with a proportionality coefficient depending on $s$ and $k$, and is $\phi$-Loewner with $\phi = \phi(s, k)$.

**Proposition** (Main statement). Let $\Omega \in \mathbb{C}_\infty$, 

$$\Omega = \mathbb{C}_\infty \setminus \bigcup_{i=1}^{n} D_i$$

where complementary components $D_i$ are $s$-relatively separated closed Jordan regions with boundaries $\partial D_i$ being $k$-quasicircles. Then $\Omega$ is a $\phi$-Loewner with $\phi = \phi(n, s, k)$.

*Proof.* By induction on $n$. Case $n = 1$ is covered by 3. For $n \geq 2$, consider arbitrary continua $E$ and $F$ in $\overline{\Omega}$ with $\Delta(E, F) \leq t$. We need to show that $\operatorname{Mod}(\Gamma(E, F; \Omega))$ is large, i.e. if a measure $\rho$ has small mass $\int_{\Omega} \rho^2 d\Sigma < m$, then it is not admissible, i.e. exists a rectifiable path $\gamma$ connecting $E$ and $F$ with $\int_{\gamma} \rho ds < 1$.

By induction hypothesis, such path already exists in the region with $n-1$ complementary components. Main idea: change the path such that it will not intersect the last component but will be still “short”. WLOG assume $D_n$ has the smallest diameter $d := \operatorname{diam}(D_n)$. There exists $m_1 = m_1(n, s, k, t) > 0$ such that if $\int \rho^2 d\Sigma < m_1$, then there exists a path $\alpha$ in $\Omega \cup D_n$ connecting $E$ and $F$ with $\int_{\alpha} \rho ds < 1/2$.

We can assume that $\alpha$ intersects $D_n$. (Otherwise the claim is obvious.)

We will remove the piece of $\alpha$ in $D_n$ and connect the remaining pieces by a path $\beta$ in $U$, where $U$ is a “Loewner collar” around $D_n$ as in Collar Lemma with $\int_{\beta} \rho ds < 1/2$ to obtain the desired path $\gamma$ in $\Omega$. Let $c = c(s, k) > 0$ be a “thickness” constant from Collar Lemma, i.e. that $N_{\frac{1}{6}cd}(D_n) \setminus D_n \subseteq U$. We will need to consider 3 cases:

**Case 1.** Neither $E$ nor $F$ is contained in $N_{\frac{1}{6}cd}(D_n)$.

Choose a closed (possibly degenerate) subpath $\alpha'$ of $\alpha$ from its endpoint $x$ on $E$ to the first point in $N_{\frac{1}{6}cd}(D_n)$, call it $x'$. Since a continuum $\alpha' \cup E$ is not contained in $N_{\frac{1}{6}cd}(D_n)$, $\alpha' \cup E \setminus B(x', r) \neq \emptyset$, where $r := \frac{1}{6} cd$. We can find a continuum $E' \subseteq \alpha' \cup E$ that is contained in $B(x', r)$ with $\operatorname{diam} E' \geq r = \frac{1}{6} cd$ (pick a connected component of $\alpha' \cup E \cap B(x', r)$ that contains $x'$). Then $E' \in U$. Likewise, we can select a subpath $\alpha''$ of $\alpha$ with endpoint on $F$ and a subcontinuum $F'$ of $F$ in $U$ with $\operatorname{diam} F' \geq \frac{1}{6} cd$. Then

$$\operatorname{dist}(E', F') \leq (2c + 1)d \leq (12 + 6/c) \min(\operatorname{diam}(E'), \operatorname{diam}(F')),$$
and thus $\Delta(E', F') \leq C(s, k)$. Since $U$ is $\phi$-Loewner with $\phi = \phi(s, k)$, there exists $m_2 = m_2(s, k) > 0$ that if $\int \rho^2 d\Sigma < m_2$, then there exists a path $\beta$ connecting $E'$ and $F'$ in $U$ with $\int_{\beta} \rho ds < 1/2$.

**Case 2.** $t \min(\text{diam}(E), \text{diam}(F)) \geq \frac{1}{3} cd$.

Choose $\alpha'$ and $\alpha''$ as in Case 1. Similarly, we can find continua $E' \subseteq \alpha' \cup E$ and $F' \subseteq \alpha'' \cup F$ with $E', F' \subseteq U$ such that $\text{diam}(E')) \geq \frac{1}{3} \min(\text{diam}(E), \text{diam}(F))$ and $\text{diam}(F')) \geq \frac{1}{3} \min(\text{diam}(F), \text{diam}(F))$. Then

$$\text{dist}(E', F') \leq (2c + 1)d,$$

$$\min(\text{diam}(E'), \text{diam}(F')) \geq \frac{1}{3} \min(\text{diam}(E), \text{diam}(F), \text{diam}(F), cd) \geq \frac{cd}{9 \max(t, 1)},$$

so $\Delta(E', F') \leq C(s, k, t)$. By Loewner property of $U$ if $\int \rho^2 d\Sigma < m_3$, where $m_3 = m_3(s, k, t) > 0$, then there is a path $\beta$ connecting $E'$ and $F'$ in $U$ with $\int_{\beta} \rho ds < 1/2$.

**Case 3.** $t \min(\text{diam}(E), \text{diam}(F)) < \frac{1}{3} cd$ and either $E$ or $F$ lies in $N_{\frac{1}{3} cd}(D_n)$.

WLOG assume $E \in N_{\frac{1}{3} cd}(D_n)$. Then $E \subseteq U$ and by the choice of $t$

$$\text{dist}(E, F) \leq t \min(\text{diam}(E), \text{diam}(F)) \leq \frac{1}{3} \text{cd}.$$

Pick $x \in E$ and $y \in F$ with $\sigma(x, y) = \text{dist}(E, F)$, and let $r = \frac{1}{3} \min(\text{diam}(F), \text{diam}(F))$. There exists (similarly to the Case 1) a continuum $F' \subseteq F \cap B(y, r)$ with $y \in F'$ and $\text{diam}(F') \geq r = \frac{1}{3} \min(\text{diam}(F), \text{diam}(F))$. Then

$$\text{dist}(E, F') = \text{dist}(E, F) \leq \frac{1}{3} \text{cd},$$

$$\text{dist}(E, F') \leq \min(t \text{diam}(E), t \text{diam}(F), \frac{1}{3} \text{cd}) \leq 3 \max(t, 1) \text{ min(} \text{diam}(E), \text{diam}(F'), \text{diam}(F)) \text{).}$$

Thus $\Delta(E, F') \leq 3 \max(t, 1)$. Since $U$ is Loewner, if $\int \rho^2 d\Sigma < m_4$, where $m_4 = m_4(s, k, t) > 0$, then there is a path $\beta$ connecting $E$ and $F'$ in $U$ with $\int_{\beta} \rho ds < 1$. Here we can simply take $\gamma = \beta$, since $\beta$ connects $E$ to $F' \subseteq F$.

Finally, we can take $m = \min\{m_1, m_2, m_3, m_4\} = m(n, s, k, t)$ and if $\int \rho^2 d\Sigma < m$, then we can find a path $\gamma$ in $\Omega$ connecting $E$ and $F$ that satisfies $\int_{\gamma} \rho ds < 1$. \(\square\)